# Construction of gravitational instantons with non-maximal volume growth

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I, Willem Adriaan Salm confirm that the work presented in my thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

## Abstract

In this thesis, we construct families of gravitational instantons of type ALG, ALG*, ALH and ALH*. Away from a finite set of exceptional points, the metric collapses with bounded curvature to a quotient of $\mathbb{R}^3$ by a lattice of rank one or two and $\mathbb{Z}_2$. Depending on whether the gravitational instantons are of type ALG/ALG* or ALH/ALH*, there are two or four exceptional points respectively that are modelled on the Atiyah-Hitchin manifold. The other exceptional points are modelled on the Taub-NUT metric. There are at most four, respectively eight, of these points in each case. These gravitational instantons are constructed using a gluing construction, where we combine these ALF gravitational instantons to a bulk space that is constructed using the Gibbons-Hawking ansatz. We then set up a deformation argument, where we perturb these approximate solutions into genuine gravitational instantons.

**Impact statement**

The study of hyperkähler manifolds has a long and rich history. It started in the 70's when Eguchi & Hanson (1978) and Yau (1978) gave the first non-trivial examples. It took however more than 40 years before Sun & Zhang (2021) classified all gravitational instantons. Due to the work of Kronheimer (1989b), Chen & Chen (2021a), Chen & Viaclovsky (2021), Chen & Chen (2021b), T. Collins et al. (2022) and Lee & Lin (2022) we now have a Torelli theorem for each type of gravitational instanton. Although all metrics are now classified, degeneration of these structures is not well understood. By the explicit nature of our construction, we can study the boundary of the moduli space in more detail.

The study of gravitational instantons is also useful in other fields. For example, according to Cherkis & Kapustin (1999), Cherkis & Kapustin (2003) and Cherkis & Ward (2012) gravitational instantons appear in gauge theory, because they arise as moduli spaces of (periodic) monopoles with Dirac singularities. Another example is due to Hawking (1979), where he explained that gravitational instantons are used in physics for understanding the quantisation of gravity.

*Unless the Lord builds the house,*
*the builders labor in vain.*

Psalm 127:1 (NIV)

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# Contents

# 1 Introduction

Gravitational instantons are examples of hyperkähler manifolds, which are Riemannian manifolds with three compatible complex structures satisfying the quaternion relations. In particular, gravitational instantons are complete, non-compact hyperkähler manifolds of dimension four, with $L^2$ bounded curvature. In the late 70's, the first constructions of gravitational instantons were found, and in 1989 Kronheimer (1989b) classified of all asymptotically locally euclidean (ALE) gravitational instantons, which have maximal volume growth. Over the years other, non-ALE, gravitational instantons were also found and recently Sun & Zhang (2021) showed that, depending on the asymptotic metric, all gravitational instantons can be classified as ALE, ALF, ALG, ALG*, ALH and ALH*. Their volume growth is of order $r^4$, $r^3$, $r^2$, $r^2$, $r$ and $r^{4/3}$ respectively. For ALG* and ALH* the curvature decay is quadratic, while in the other classes it is faster than quadratic.

In this thesis, we propose a new construction for gravitational instantons for the classes ALG, ALG*, ALH and ALH*. To construct these, we will use a gluing method in geometry, which was pioneered by the work of Taubes (1982) in gauge theory. Namely, we start with a non-complete hyperkähler manifold as our bulk space and combine it with other gravitational instantons using the connected sum construction. After interpolating the metrics we get an approximate solution, and by a perturbation argument we make it into a genuine hyperkähler manifold. The idea for which spaces to glue in is due to Sen (1997). He proposed to use $n$ copies of Taub-NUT and one copy of the Atiyah-Hitchin space as 'bubbles' and use the construction by Gibbons & Hawking (1978) to create the 'bulk space'. This proposal was carried out rigorously by Schroers & Singer (2021).

Recall that the Gibbons-Hawking ansatz is a general construction for hyperkähler metrics on circle bundles over 3-dimensional manifolds with a triple of parallel, orthonormal vector fields. In Sen's construction, the 3-dimensional manifold is $\mathbb{R}^3 \setminus \{0, \pm p_1, \ldots, \pm p_n\}$ for some distinct $p_i \neq 0$. This gluing construction can be done with base spaces other than subsets of $\mathbb{R}^3$. For example, Foscolo (2019) used Sen's method on a punctured $T^3$ in order to construct hyperkähler metrics on the K3 surface. In this thesis, we will consider similar cases: We will apply Sen's method on $\mathbb{R}^3$ modulo a lattice of rank one or two. Although this looks like a minor change, the analysis changes dramatically and we have to study each situation separately.

The statement of our main theorem is as follows:

**Theorem 1.1.** *Let $L \subset \mathbb{R}^3$ be a lattice of rank one or two and consider the $\mathbb{Z}_2$ action on $\mathbb{R}^3/L$ that is induced by the antipodal map on $\mathbb{R}^3$. Let $\{p_i\}$ be a configuration of $n$ distinct points in $(\mathbb{R}^3/L - \mathrm{Fix}(\mathbb{Z}_2))/\mathbb{Z}_2$. Suppose that $n \leq 4$ when $\mathbb{R}^3/L \simeq \mathbb{R}^2 \times S^1$ and $n \leq 8$ when $\mathbb{R}^3/L \simeq \mathbb{R} \times T^2$. Then, there exists an $\epsilon_0 > 0$, such that for all $0 < \epsilon < \epsilon_0$ there exist a gravitational instanton $(M_{\mathbb{R}^3/L,n}, g_\epsilon)$ with the following properties:*

1. *For each fixed point of the $\mathbb{Z}_2$ action on $\mathbb{R}^3/L$, there is a compact set $K \subset M_{\mathbb{R}^3/L,n}$, such that $\epsilon^{-2}g_\epsilon$ approximates the Atiyah-Hitchin metric on $K$ as $\epsilon \to 0$.*

2. *For each $i \in \{1, \ldots, n\}$, there is a compact set $K_i \subset M_{\mathbb{R}^3/L,n}$ such that $\epsilon^{-2}g_\epsilon$ approximates the Taub-NUT metric on $K_i$ as $\epsilon \to 0$.*

3. *Away from the singularities, the manifold collapses to $(\mathbb{R}^3/L)/\mathbb{Z}_2$ with bounded curvature as $\epsilon$ converges to zero.*

4. *Depending on the lattice and $n$, the asymptotic metric can be classified as*

    - *ALG*-$I^*_{4-n}$ when $\dim L = 1$ and $n < 4$,*
    - *ALG$_{\frac{1}{2}}$ when $\dim L = 1$ and $n = 4$,*
    - *ALH*$_{8-n}$ when $\dim L = 2$ and $n < 8$,*
    - *ALH when $\dim L = 2$ and $n = 8$.*

The explicit definitions of ALG, ALG*, ALH and ALH* will be given in Section 3.5, where we will compare our metrics with existing literature. In order to examine the similarities and differences between our work and Schroers & Singer (2021), we also consider the case $L = \{0\}$ and put their work in the same systematic framework as the new cases in Theorem 1.1.

### Overview of the chapters

In the first part of Chapter 2, we introduce gravitational instantons and give some basic properties. We explain the different types of gravitational instantons and we give several examples of their construction. In order to describe the asymptotic metric, we explain the construction by Gibbons and Hawking.

In our examples we will focus especially on the Taub-NUT space and the Atiyah-Hitchin manifold. Furthermore, we explain the gauge-theoretic definition of the Atiyah-Hitchin manifold and how the asymptotic metric on the branched double cover relates to a Taub-NUT metric with negative mass. We also revisit its topology and the different circle fibrations the Atiyah-Hitchin manifold possesses.

In the second part of Chapter 2, we explain the gluing construction. First we give a general overview of the main steps. Secondly, we show how the hyperkähler property can be formulated in terms of orthonormal triples of closed self-dual 2-forms. Using this description, we setup a perturbation problem and we demonstrate how the hyperkähler condition can be rephrased as an elliptic equation. We claim that up to a small error, the linearised version of this elliptic equation is just the Laplacian on functions and this Laplacian will be the main focus in our analysis.

In Chapter 3 we construct the underlying manifold and equip it with an approximate hyperkähler metric satisfying the required asymptotics at infinity. In Section 3.1 we construct the bulk space using the Gibbons-Hawking ansatz and we show it has the properties we require. In Section 3.2 we deviate from the main topic and consider the degrees of freedom we have in our construction of the bulk space. In Section 3.3 we return to the main topic and introduce the collapsing parameter by which we can control the error. In Section 3.4 we apply the connected sum construction. We show the metrics can be interpolated in such way that we can apply the perturbation method described in Chapter 2. Moreover, we explain how the asymptotic behaviour of the bulk space gives us an error estimate for the approximate solution.

In Section 3.5 we calculate the topology of our manifolds and explain the argument by Sen (1997) to show that the intersection matrix of an ALF-gravitational instanton produced by his suggested gluing construction, is the Cartan matrix of a $D_k$-Dynkin diagram. We extend his argument and show how the intersection matrices for our

ALG/ALG* gravitational instantons relate to the extended $D_k$-Dynkin diagrams. We also calculate the homology of our ALH/ALH* gravitational instantons. We compare our results to the known classifications of gravitational instantons and we conclude that our degrees of freedom coincide with the dimensions of the moduli spaces for each type of gravitational instanton. By calculating the monodromy at infinity, we show that our ALG/ALG* gravitational instantons can be compactified by adding an elliptic $I_k^*$-fiber. Similarly we show that our ALH/ALH* gravitational instantons can be compactified by adding an elliptic $I_k$-fiber.

In chapter 4 we set up the weighted analysis on the asymptotic region of the complete, almost hyperkähler manifold using the structure of the Gibbons-Hawking ansatz. Despite the different asymptotic structures, we give an approach that is uniform for ALF, ALG, ALG*, ALH and ALH*. We do this by finding the correct conformal rescaling such that the universal cover over a fixed set of charts has bounded geometry. We rephrase our linearised operator in terms of a weighted operator and we show that it is bounded and strictly elliptic.

In Section 4.2 we give a general method to convert a standard elliptic estimate on $\mathbb{R}^n$ to a local elliptic estimate in our weighted spaces. By applying this framework, we get all the local elliptic regularity estimates we need. In Section 4.3 we combine these local estimates into estimates on the whole asymptotic region. Using the Poincaré inequality we improve these results so that they imply the Laplacian is Fredholm. We use these results in Section 4.4 and calculate the kernel and co-kernel explicitly for small weights. We see that for our ALF gravitational instantons there is a certain range of weights where the operator is bijective. For the other cases we have an index of $\pm 1$ for the weights we are interested in. Using the explicit description of the kernel, we manually change the domain, making the Laplacian an isomorphism.

In Chapter 5 we finalise the analysis and give the main proof of the theorem. In Section 5.1 we extend our weighted norms on the asymptotic region to the whole space and show that the Laplacian is still a bounded, strictly elliptic operator. Although any compact extension will yield this result, extra care is taken in order to apply the bounded inverse estimate in Section 5.4. In Sections 5.2 and 5.3 we extend our elliptic estimates globally and we show that the Laplacian is still Fredholm. In Section 5.2 we focus on the Sobolev norm, and using our understanding of the (co)-

kernel on the asymptotic region we show the bijectivity of the Laplacian between certain weighted spaces. In Section 5.3 we show that the same result holds for Hölder spaces. In Section 5.5 we finalise the proof of the main theorem. We set up the Banach spaces on which we do the perturbation argument and we show that the conditions for the inverse function theorem are satisfied. We also show that the linearised operator in this perturbation argument indeed approximates the standard Laplacian on functions and hence it is also invertible with a bounded inverse. This proves our main theorem.

# 2 Background

In this thesis we propose a new method of constructing gravitational instantons. Before we delve into the construction, we first explain what gravitational instantons are, list some main properties and give some examples. We focus on examples constructed via the Gibbons-Hawking ansatz and on the Atiyah-Hitchin manifold, because these will be fundamental building blocks in our new construction. Also, the Gibbons-Hawking ansatz is used to describe the asymptotic structure of many gravitational instantons and is needed to understand the classification by Sun & Zhang (2021).

In the second part of this chapter we give a pictorial explanation of our construction. We explain the gluing construction for gravitational instantons and show how to turn this gluing problem into an elliptic equation. In later chapters we set up the analysis to solve this equation.

## 2.1 Gravitational instantons

We use the following definitions of a hyperkähler manifold and of a gravitational instanton:

**Definition 2.1.** *A hyperkähler manifold* $(M, g, I, J, K)$ *is a* $4n$*-dimensional, Riemannian manifold* $(M, g)$ *with three integrable complex structures* $I$*,* $J$ *and* $K$ *such that*

- $I$*,* $J$*, and* $K$ *are Kähler with respect to* $g$*, and*
- $I$*,* $J$*, and* $K$ *satisfy the quaternion relations*

$$\begin{aligned} I \cdot J &= K, & J \cdot K &= I, & K \cdot I &= J, \\ J \cdot I &= -K, & K \cdot J &= -I, & I \cdot K &= -J. \end{aligned}$$

**Definition 2.2.** *A gravitational instanton is a* $4$*-dimensional, complete, non-compact hyperkähler manifold* $(M, g, I, J, K)$ *such that the Riemann curvature tensor* $\mathrm{Rm}$ *is* $L^2$*-bounded.*

Hyperkähler manifolds are already interesting in their own right. For example, they show up as a special case in the classification of Riemannian manifolds by Berger (1953): He showed that the holonomy of a simply connected, irreducible and non-symmetric Riemannian manifold can only be one of $SO(n)$, $U(n)$, $SU(n)$, $Sp(n) \cdot Sp(1)$, $Sp(n)$, $G_2$ or $Spin(7)$. The class corresponding the compact symplectic group $Sp(n)$ corresponds to hyperkähler metrics. This is because $Sp(n)$ can be viewed as the set of all $n \times n$ matrices with entries in $\mathbb{H}$ that preserve the standard hermitian inner product on $\mathbb{H}^n$. From this identification of $\mathbb{H}^n$ we get three almost complex structures $I$, $J$ and $K$ on $M$, which must be compatible with the metric $g$. These complex structures are parallel and hence the Riemannian manifold must be hyperkähler.

Because $Sp(n)$ is a subgroup of $SU(2n)$, Hyperkähler manifolds are examples of Calabi-Yau manifolds. These are Kähler manifolds $(M, g, I)$ with a non-vanishing, parallel, holomorphic volume form $\Omega$. For hyperkähler manifolds this volume form can be explicitly identified. Namely, if $\omega_J$ and $\omega_K$ are the Kähler forms corresponding to $J$ and $K$, then one can choose $\Omega = (\omega_J + i\omega_K)^n$ as the holomorphic volume form. For dimension 4, $Sp(1) = SU(2)$, and so 4-real-dimensional hyperkähler manifolds are the same as 4-real-dimensional Calabi-Yau manifolds.

In four dimensions, one can decompose $\Omega^2$ into the self- and anti-self-dual forms and one can write the Riemann curvature operator as

$$\mathrm{Rm} = \begin{pmatrix} W^+ + \frac{\mathrm{Scal}}{12} & \mathring{\mathrm{Ric}} \\ \mathring{\mathrm{Ric}} & W^- + \frac{\mathrm{Scal}}{12} \end{pmatrix},$$

where Scal is the Scalar curvature and $\mathring{\mathrm{Ric}}$ is the traceless Ricci curvature. For Calabi-Yau manifolds, the Ricci tensor vanishes, so the traceless Ricci curvature and the scalar curvature vanishes too. Furthermore, the self-dual part of the Weyl tensor is also zero, because $\Lambda^+$ is trivialised by the three (parallel) Kähler forms. Therefore, the Riemann curvature tensor is anti-self-dual. Using the extra condition that $\int |\mathrm{Rm}|^2$ is bounded, one can view gravitational instantons as minimisers of the Yang-Mills like action $g \mapsto \int \mathrm{Rm} \wedge * \mathrm{Rm}$. Because of this similarity with the Yang-Mills instantons, complete 4-dimensional hyperkähler manifolds with $L^2$ bounded curvature are called gravitational instantons.

### The Gibbons-Hawking ansatz

In order to give some non-trivial examples of gravitational instantons, we explain the construction by Gibbons & Hawking (1978). Their construction starts with the following information:

1. An open subset $U$ in $\mathbb{R}^3$, equipped with the Euclidean metric $g_U$,
2. a principal $S^1$-bundle $P$ over $U$,
3. a connection $\eta$ on $P$, and
4. a harmonic function[1] $h\colon U \to (0,\infty)$ satisfying the Bogomolny equation

$$*^{g_U} \mathrm{d}\, h = \mathrm{d}\, \eta.$$

In their paper, Gibbons and Hawking show

**Proposition 2.3.** *The metric $g^{GH} = h\, g_U + h^{-1}\eta^2$ on $P$ is hyperkähler.*

This construction is called the Gibbons-Hawking ansatz. Some of the requirements above are redundant. Namely, if $U$ is an open subset of $\mathbb{R}^3$ and $h$ is a positive harmonic function, then $* \mathrm{d}\, h$ is closed and $[* \mathrm{d}\, h]$ is an element in de Rham cohomology. We claim that if $[* \mathrm{d}\, h] \in H^2(U,\mathbb{Z})$, we can construct the principal bundle $P$ and find a connection $\eta$ satisfying the Bogomolny equation. Indeed, all circle bundles are uniquely classified by the first Chern class, and hence $P$ is uniquely determined by $c_1 = [* \mathrm{d}\, h]$. Moreover, every principal bundle admits a connection $\eta$ and by adding some element of $\Omega^1(U)$, we can always assume that $\mathrm{d}\, \eta = * \mathrm{d}\, h$. In summary, for the Gibbons-Hawking ansatz we only need

1. an open subset $U$ in $\mathbb{R}^3$, and
2. a positive harmonic function that satisfies $[* \mathrm{d}\, h] \in H^2(U,\mathbb{Z})$.

*Proof of Proposition 2.3.* In order to define the complex structures on $P$, it is sufficient to define the Kähler forms. Equipping $\mathbb{R}^3$ with the standard coordinates $\{x_i\}$, we define

$$\omega_i = \mathrm{d}\, x_i \wedge \eta + h *^{g_U} \mathrm{d}\, x_i$$

[1] i.e. $\Delta h = 0$ where $\Delta$ is the Laplacian.

For example $\omega_1 = \mathrm{d}\, x_1 \wedge \eta + h \,\mathrm{d}\, x_2 \wedge \mathrm{d}\, x_3$. Each form $\omega_i$ defines an endomorphism $I_i$ on $TM$ such that $I_i^* \eta = h \,\mathrm{d}\, x_i$ and

$$I_i^* \,\mathrm{d}\, x_j = \begin{cases} \frac{-1}{h}\eta & i = j \\ -\operatorname{sgn}(ijk)\,\mathrm{d}\, x_k & i \neq j. \end{cases}$$

Therefore $I_i$ are almost complex structures on $P$ that satisfy the quaternion relations.

To show that these complex structures are integrable, we need to show that the exterior derivative is a map between $\Omega^{1,0}$ and $\Omega^{2,0} \oplus \Omega^{1,1}$. Hence for any $\alpha \in \Omega^{1,0}$ and $\beta \in \Omega^{2,0}$, we need to show $\langle \mathrm{d}\, \alpha, \bar{\beta} \rangle = 0$. We claim that $\omega_{\mathbb{C}} = \omega_j + i\omega_k$ is a basis for $\Omega^{2,0}$, because for any $X \in \Gamma(\Lambda^{0,1}TP)$,

$$\omega_j(X, \ldots) + i\omega_k(X, \ldots) = g(JX, \ldots) - ig(JIX, \ldots) = g(JX, \ldots) - g(JX, \ldots) = 0.$$

With respect to the metric $g^{GH}$, the 2-forms $\omega_i$ are self dual and so $\langle \mathrm{d}\, \alpha, \bar{\omega}_{\mathbb{C}} \rangle = \mathrm{d}\, \alpha \wedge \omega_{\mathbb{C}}$. By the Bogomolny equation, the 2-forms $\omega_i$ are closed and hence $\langle \mathrm{d}\, \alpha, \bar{\omega}_{\mathbb{C}} \rangle = \mathrm{d}(\alpha \wedge \omega_{\mathbb{C}})$. This vanishes, because $\alpha \wedge \omega_{\mathbb{C}} \in \Omega^{3,0}$ on a 2-complex-dimensional manifold. $\square$

The simplest example of the Gibbons-Hawking ansatz is when $U = \mathbb{R}^3$ and $h$ is constant. The first Chern class $h$ induces is zero and so the Gibbons-Hawking ansatz gives a flat hyperkähler metric on $\mathbb{R}^3 \times S^1$. From the formula $g^{GH} = h \cdot g_U + h^{-1}\eta^2$, it follows that the circle radius is inversely proportional to the constant value of $h$.

### ALE-type gravitational instantons

The next example we explain is for $U = \mathbb{R}^3 \setminus \{0\}$ and $h(x) = \frac{k}{2|x|}$. In order to confirm $[* \,\mathrm{d}\, h] \in H^2(U, \mathbb{Z})$, we determine when $-\frac{1}{2\pi}\int_{S^2} * \,\mathrm{d}\, h$ is an integer[2]. The factor in the denominator in $h$ is chosen such that $-\frac{1}{2\pi}\int_{S^2} * \,\mathrm{d}\, h$ is exactly $k$ and hence $[* \,\mathrm{d}\, h] \in H^2(U, \mathbb{Z})$ if and only if $k \in \mathbb{Z}$. Notice that $* \,\mathrm{d}\, h = \frac{k}{2} \mathrm{Vol}_{S^2}$ does not depend on the radial parameter in $\mathbb{R}^3$. Hence $P$ is diffeomorphic to $\mathbb{R}^+$ times a degree $k$ circle bundle over $S^2$.

The Gibbons-Hawking metric for this case is $g^{GH} = \frac{k}{2r}(\mathrm{d}\, r^2 + r^2 g_{S^2}) + \frac{2r}{k}\eta^2$. If we

[2] The sign is due to the identification of $\mathbb{R}$ with $\mathfrak{u}(1)$.

reparametrise this by $s = \sqrt{2kr}$, then

$$g^{GH} = \mathrm{d}\, s^2 + s^2 \left( \frac{1}{4} g_{S^2} + \frac{1}{k^2} \eta^2 \right).$$

The part between brackets is an $s$-invariant metric on the circle bundle over $S^2$. By identifying this circle bundle with the Hopf projection $p \mapsto pi\bar{p}$ from $S^3 \subset \mathbb{H}$ to $S^2 \subset \operatorname{im} \mathbb{H}$, quotiented by the $\mathbb{Z}_k$ action $p \sim p \cdot e^{\frac{2\pi i}{k}}$, one concludes that $\frac{1}{4} g_{S^2} + \frac{1}{k^2}\eta^2$ is the standard metric on $S^3/\mathbb{Z}_k$. Therefore, $g^{GH}$ is just the flat metric on $\mathbb{R}^4/\mathbb{Z}_k$. This metric is not complete, but can be completed by adding a single point when $k = 1$.

A nice feature of the Gibbons-Hawking ansatz is its additive property: Given two positive harmonic functions $h_1, h_2 \colon U \to \mathbb{R}$, both inducing integral cohomology classes, their sum does too. For example, pick $x_0 \in \mathbb{R}^3 \setminus \{0\}$ and consider the positive harmonic function $h_1(x) = \frac{1}{2|x|}$ on $U = \mathbb{R}^3 \setminus \{0, x_0\}$. By Stoke's theorem one can check $* \mathrm{d}\, h_1$ induces an integral cohomology class on $U$. By the rotation and translation invariance of the Laplacian $h_2(x) = \frac{1}{2|x - x_0|}$ is also a positive harmonic function and it also induces an integral cohomology class on $U$. Because cohomology classes and the space of harmonic functions are both $\mathbb{Z}$-linear, $h(x) = \frac{1}{2|x|} + \frac{1}{2|x-x_0|}$ is a positive harmonic function and $[* \mathrm{d}\, h] \in H^2(U, \mathbb{Z})$. Therefore, we can apply the Gibbons-Hawking Ansatz on $h$ and the Gibbons-Hawking metric is

$$g^{GH} = \left( \frac{1}{2|x|} + \frac{1}{2|x - x_0|} \right) g_U + \left( \frac{1}{2|x|} + \frac{1}{2|x - x_0|} \right)^{-1} \eta^2.$$

Just as in the previous example, we can complete this space by adding points. At the boundary at infinity, the manifold decomposes as a radial parameter and a degree 2 circle bundle over $S^2$. Here the metric approximates the flat metric on $\mathbb{R}^4/\mathbb{Z}_2$, because for large values of $x$, $\frac{1}{2|x|} + \frac{1}{2|x-x_0|} \simeq \frac{2}{2|x|}$.

According to Prasad (1979), this metric is the Eguchi-Hanson metric on $T^*S^2$. To see the generator of $H_2(T^*S^2)$, one considers a path between 0 and $x_0$. Outside the singularities, the total space retracts to a trivial circle bundle over the open path, which is topologically a cylinder. At the endpoints the harmonic function $h$ diverges to infinity, which has the effect that the circle radius collapses at these points. This creates a sphere that generates $H_2(T^*S^2)$.

There is nothing stopping us from repeating the procedure in the last example. The Gibbons-Hawking metric related to $h = \sum_{i=1}^{k} \frac{1}{2|x-p_i|}$ for some distinct $p_1, \dots, p_k \in \mathbb{R}^3$ yields a gravitational instanton that near infinity approximates the flat metric on $\mathbb{R}^4/\mathbb{Z}_k$. Moreover, it retracts to a chain of wedge sums of $k-1$ two-spheres. In the literature these spaces are called multi-Eguchi-Hanson spaces.

All the examples above have the property that up to some small error they approximate the flat metric on $\mathbb{R}^4$ near infinity. These examples are part of a single class of gravitational instantons called ALE:

**Definition 2.4.** *A gravitational instanton $(M, g)$ is called Asymptotically Locally Euclidean (ALE) if there is a finite subgroup $\Gamma$ of $SU(2)$, two large compact sets $K_1 \subset M$ and $K_2 \subset \mathbb{C}^2/\Gamma$ and a diffeomorphism $\varphi$ between $M \setminus K_1$ and $(\mathbb{C}^2/\Gamma) \setminus K_2$ such that on these asymptotic regions*

$$\|\nabla^k(\varphi_* g - g_{\mathbb{C}^2/\Gamma})\|_{g_{\mathbb{C}^2/\Gamma}} = \mathcal{O}(r^{-4-k})$$

*for all $k \in \mathbb{N}$.*

All ALE gravitational instantons were constructed by Kronheimer (1989a). His method makes use of the hyperkähler quotient construction. This is a generalisation of the Kähler quotient for hyperkähler manifolds. Namely, consider a hyperkähler manifold $(M, g)$ and a Lie group $G$ that acts preserving on the triple of Kähler forms $\omega_1$, $\omega_2$ and $\omega_3$. Also assume that each Kähler form $\omega_i$ has a moment map $\mu_i \colon M \to \mathfrak{g}^*$. That is, $\mu_i \colon M \to \mathfrak{g}^*$ is a $G$-equivariant function such that for each $\xi \in \mathfrak{g}$, the vector field $X_\xi$ generated by $\xi$ satisfies $\iota_{X_\xi}\omega_i = \mathrm{d}\langle \mu_i, \xi \rangle$. Just as the Kähler quotient, for each regular value $\eta$ of $\mu := (\mu_1, \mu_2, \mu_2) \colon M \to \mathfrak{g}^* \otimes \mathbb{R}^3$, the space $\mu^{-1}(\eta)/G$ is a smooth manifold and $(g, \omega_1, \omega_2, \omega_3)$ descends to a hyperkähler structure on the quotient.

Kronheimer started with a finite subgroup $\Gamma$ of $SU(2)$. Given this group he considered the finite dimensional Hilbert space $R = L^2(\Gamma)$, its unitary group $U(R)$ and its Lie algebra $\mathfrak{u}(R)$. Inside $\mathfrak{u}(R) \otimes \mathbb{H}$ he considered the linear subspace $(\mathfrak{u}(R) \otimes \mathbb{H})^\Gamma$ of all elements that are invariant under the $\Gamma$ action and he used this as the flat hyperkähler manifold on which he applied the hyperkähler quotient construction. The group $U(R)$ acts on $\mathfrak{u}(R) \otimes \mathbb{H}$ by conjugation and its stabilizer is the subgroup consisting of scalar multiplication and therefore, $PU(R) := U(R)/S^1$ acts freely on $\mathfrak{u}(R)\otimes\mathbb{H}$. Inside $PU(R)$, Kronheimer considered the subgroup of all invariant elements that commute with $\Gamma$ as the quotient group. For any $A = A_0+iA_1+jA_2+kA_3 \in (\mathfrak{u}(R)\otimes\mathbb{H})^\Gamma$ he used the moment maps

$$\begin{aligned}\mu_1(A) =&[A_0, A_1] + [A_2, A_3],\\ \mu_2(A) =&[A_0, A_2] + [A_3, A_1],\\ \mu_3(A) =&[A_0, A_3] + [A_1, A_2].\end{aligned}$$

This way he constructed a hyperkähler metric on the minimal resolution $\widetilde{\mathbb{C}^2/\Gamma}$ of $\mathbb{C}^2/\Gamma$. By studying the irreducible representations of $(\mathfrak{u}\otimes\mathbb{H})^\Gamma$ and using the McKay correspondence, Kronheimer showed that these gravitational instantons can be realised as a quiver variety, where the quiver is the affine Dynkin diagram associated to $\Gamma$.

Kronheimer (1989b) also classified all ALE gravitational instantons and showed that the construction in Kronheimer (1989a) produces all examples of ALE gravitational instantons. Namely, up to some non-degeneracy condition, there is a hyperkähler structure $(g, \omega_1, \omega_2, \omega_3)$ for every $(\sigma_1, \sigma_2, \sigma_3) \in H^2(\widetilde{\mathbb{C}^2/\Gamma}, \mathbb{R}) \otimes \mathbb{R}^3$ such that $[\omega_i] = \sigma_i$. This hyperkähler structure is unique up to tri-holomorphic isomorphisms. The necessity of the non-degeneracy condition can already be seen for the Eguchi-Hanson metric. Namely, when the singularity $x_0$ tends to 0, the volume of $S^2$ collapses and the metric becomes the singular metric of $\mathbb{R}^4/\mathbb{Z}_2$. To exclude this case, one has to describe $x_0$ using topological terms. This can be done by integrating the Kähler forms over $S^2$. Indeed, except for the endpoints, the 2-sphere can be viewed as the trivial circle bundle over the straight line between 0 and $x_0$. Hence,

$$x_0^i = \frac{1}{2\pi}\int_{S^2} \mathrm{d}\, x_i \wedge \eta = \frac{1}{2\pi}\int_{S^2} \omega_i = \sigma_i(S^2).$$

In general one requires that $(\sigma_1(\Sigma), \sigma_2(\Sigma), \sigma_3(\Sigma)) \neq 0$ for all $\Sigma$ in $H_2(\widetilde{\mathbb{C}^2/\Gamma}, \mathbb{Z})$ with self-intersection -2.

*Remark* 2.5. The classification result by Kronheimer is an example of a Torelli theorem. In these theorems one identifies all metrics up to triholomorphic isometries in terms of their model at infinity and their periods, i.e. the integrals of the Kähler forms over the 2-cycles with self-intersection -2. In general, one also has a non-degeneracy condition, as in the ALE case.

### ALF-type gravitational instantons

Next we will give an example of the Gibbons-Hawking ansatz that is not ALE. For this we consider the harmonic function $c + \frac{1}{2r}$ for some constant $c > 0$. By the additive property of the Gibbons-Hawking ansatz, this is a hyperkähler manifold and, as before, it can be completed into a gravitational instanton by adding a single point. The Gibbons-Hawking metric is

$$g^{GH} = (c + (2r)^{-1})g_{\mathbb{R}^3\setminus\{0\}} + \frac{1}{c + (2r)^{-1}}\eta^2.$$

Near infinity the metric approximates the cylindrical metric $c \cdot g_{R^3\setminus\{0\}} + \frac{1}{c}\eta^2$ on $\mathbb{R}^+ \times S^3$. The value of $c^{-\frac{1}{2}}$ is proportional to the circle radius of a fiber. This gravitational instanton cannot be ALE, because the circle radius of a fiber does not grow linearly but converges to a constant. Also, the volume growth of a ball of radius $r$ is of order $r^3$ instead of $r^4$. This metric was found by Taub and extended by Newman, Unti and Tamburino. It is called the Taub-NUT space and it is an example of an ALF-type gravitational instanton.

**Definition 2.6.** *A gravitational instanton $(M, g)$ is of type ALF-$A_k$ if there are $\epsilon, c > 0$, $k \in \mathbb{Z}_{>0}$ and a diffeomorphism $\varphi$ between the asymptotic regions of $M$ and the Gibbons-Hawking space for $h(x) = c + \frac{k}{2|x|}$ on $\mathbb{R}^3 \setminus \{0\}$ such that*

$$\|\nabla^j(\varphi_* g - g^{GH})\|_{g^{GH}} = \mathcal{O}(r^{-j-\epsilon})$$

*for all $j \in \mathbb{N}$. If instead there is a diffeomorphism between the asymptotic regions of $M$ and a $\mathbb{Z}_2$ quotient of the Gibbons-Hawking space for $h = c + \frac{2k-4}{2|x|}$ and this $\mathbb{Z}_2$ quotient is a lift of the antipodal map on $\mathbb{R}^3$, then we say the gravitational instanton $(M, g)$ is of type ALF-$D_k$. In general we call a gravitational instanton Asymptotically Locally Flat (ALF) if it is of type ALF-$A_k$ or ALF-$D_k$.*

All ALF-$A_k$ gravitational instantons are classified by Minerbe (2011): They all arise by the Gibbons-Hawking ansatz for $h = c + \sum_{i=1}^{n} \frac{1}{2|x-p_i|}$ for some set of distinct points $p_i \in \mathbb{R}^3$. These spaces are also known as multi-Taub-NUT spaces. Similarly to the multi-Eguchi-Hanson space, the multi-Taub-NUT space retracts to a chain of wedge sums of 2-spheres. The intersection matrix is the negative Cartan matrix for an $A_k$-Dynkin diagram. A similar result is true for ALF-$D_k$ type gravitational instantons.

A famous example of an ALF-$D_0$ gravitational instanton is the Atiyah-Hitchin manifold. It is the moduli space of centred magnetic monopoles of charge 2. To understand this, we consider an $SU(2)$-connection $A$ over $\mathbb{R}^3$ and a Higgs field $\phi\colon \mathbb{R}^3 \to \mathfrak{su}(2)$ that minimises the Yang-Mills-Higgs energy $\int_{\mathbb{R}^3} \|F_A\|^2 + \|D_A\phi\|^2$. By assuming $|F_A| = \mathcal{O}(r^{-2})$ and $|\phi| = 1 + \frac{k}{2r} + \mathcal{O}(r^{-2})$ for some $k \in \mathbb{N}$, we force the energy to be finite. With these decay conditions, the energy functional can be rewritten as

$$\int_{\mathbb{R}^3} \|F_A\|^2 + \|D_A\phi\|^2 = \int_{\mathbb{R}^3} \|F_A - *D_A\phi\|^2 \pm 8\pi k$$

and hence the pair $(A, \phi)$ minimises the Yang-Mills-Higgs energy if and only if it satisfies the Bogomolny equation $F_A = *D_A\phi$. The space of pairs $(A, \phi)$ form an infinite dimensional space, however the gauge orbits have finite co-dimension. The moduli space $N_k$ of magnetic monopoles of charge $k$ is the quotient of the space of pairs $(A, \phi)$ and the group of all gauge transformations and it is $(4k-1)$-dimensional.

The energy functional is invariant under translations on $\mathbb{R}^3$. The quotient of $N_k$ by the translation action is called the moduli space of centred magnetic monopoles of charge $k$. This space, which we denote by $M_k^0$ is $(4k-4)$-dimensional and it is complete. In our study of gravitational instantons we only focus on the 4-dimensional manifolds and so we define the Atiyah-Hitchin manifold as $M_2^0$.

The metric on the Atiyah-Hitchin manifold can be defined using an infinite dimensional version of the hyperkähler quotient construction. Namely, pairs $(A, \phi)\colon \mathbb{R}^3 \to \mathfrak{su}(2) \otimes \mathbb{R}^4$ form an infinite dimensional vector space and one can equip this with a quaternionic structure by identifying $(A_i \,\mathrm{d}\, x_i, \phi)$ as $\phi + A_1\, I + A_2\, J + A_3\, K$. The three components of the Bogomolny equation $F_A - *D_A\phi$ can be viewed as moment maps for the group of all gauge transformations $\mathcal{G}$. In order to make sense of the quotient $\mu^{-1}(0)/\mathcal{G}$ between infinite-dimensional spaces, one has to study the deformation problem

$$\Omega^0(\mathfrak{g}) \xrightarrow{\mathrm{d}_1} \Omega^1(\mathfrak{g}) \oplus \Omega^0(\mathfrak{g}) \xrightarrow{\mathrm{d}_2} \Omega^2(\mathfrak{g})$$

where $d_2$ is the linearized Bogomolny equation and $d_1$ arises from the infinitesimal gauge transformations. With the requirement that variations of $(A, \phi)$ are square integrable, this becomes a Fredholm complex. After we restrict $\mathcal{G}$ to the group $\mathcal{G}'$ of gauge transformations whose Lie-algebra elements decay with order $r^{-1}$, the quotient $M_k := \cap_i \mu_i(0)/\mathcal{G}'$ becomes a $4k$-dimensional hyperkähler manifold.

The group $\mathcal{G}'$ is a proper subgroup of $\mathcal{G}$ and it turns out that $M_k$ is a circle bundle over $N_k$. Moreover, on the universal cover, the translation and $S^1$ action span a flat quaternionic space. Therefore, the centred moduli space $M_k^0 = \frac{M_k}{S^1 \times \mathbb{R}^3}$ has a hyperkähler structure.

The rotations of $\mathbb{R}^3$ induce an isometric action of $SU(2)$ on $M_k^0$ that rotates the complex structures. This enabled M. Atiyah & Hitchin (1988) to write their metric in the form

$$g^{AH} = (abc)^2 \,\mathrm{d}\, s^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2,$$

where $a, b, c$ are functions on the Atiyah-Hitchin manifold and $\sigma_i$ is the basis of the left invariant 1-forms. They solved this explicitly in terms of elliptic integrals and gave its asymptotic expansion: On the branched double cover at infinity this metric approximates the Taub-NUT metric with negative mass $-4$

$$g^{TN'} := (1 - 2/r)(\mathrm{d}\, r^2 + r^2 g_{S^2}) + \frac{1}{1 - 2/r}\eta^2$$

up to exponentially small terms.

**Topology and orbits of the Atiyah-Hitchin manifold**

To explain the orbits induced by the rotation action on $\mathbb{R}^3$ and the topology of the Atiyah-Hitchin manifold, we refer to the work of Schroers & Singer (2021). In their appendix they studied $\mathbb{C}P^1 \times \mathbb{C}P^1$ with the $SU(2)$ action acting diagonally. Inside they identified two copies of $\mathbb{C}P^1$, calling them the diagonal and anti-diagonal. They define the Atiyah-Hitchin manifold as the complement of the anti-diagonal, quotiented by some $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. We will explain these things in more detail.

We consider $\mathbb{C}P^1$ as the space of unit quaternions quotiented by the left multiplication of the circle action $e^{i\phi}$. Using the Hopf projection $p \mapsto \bar{p}ip$, one can identify an element of $\mathbb{C}P^1 \times \mathbb{C}P^1$ as $(\bar{q}iq, \bar{p}ip)$ for some $p, q \in SU(2)$. In this representation, the diagonal $SU(2)$-action acts as $(\bar{q}iq, \bar{p}ip) \xmapsto{r \in SU(2)} (\bar{r}\, \bar{q}iq\, r, \bar{r}\, \bar{p}ip\, r)$. We claim that each orbit for this diagonal $SU(2)$ action has a unique representative of the form $(ie^{j\phi}, i)$ for some $\phi \in [0, \pi/2]$. Indeed, pick an element $(\bar{q}iq, \bar{p}ip) \in \mathbb{C}P^1 \times \mathbb{C}P^1$. For any $\theta \in \mathbb{R}$, we can use the diagonal $SU(2)$ action to perform the transformation $(\bar{q}iq, \bar{p}ip) \xmapsto{\bar{p}e^{i\theta}} (e^{-i\theta}p \cdot \bar{q}iq \cdot \bar{p}e^{i\theta}, i)$. Because $q\bar{p}$ is a unit quaternion, $p\bar{q}iq\bar{p} \in \operatorname{im}\mathbb{H}$, and so $p\bar{q}\, i\, q\bar{p}$ can be written as $ix + vk$ for some $x \in [-1, 1]$ and $v \in \mathbb{C}$. Next we choose $\theta$ such that $v$ is a non-negative real number. In polar coordinates this simplifies to $(ie^{2j\phi}, i)$, which proves our claim. In general, an element inside $\mathbb{C}P^1 \times \mathbb{C}P^1$ can be written as $(\bar{p}ie^{2j\phi}p, \bar{p}ip)$ for some $p \in SU(2)$ and $\phi \in [0, \pi/2]$.

When $\phi = 0$, all elements in the orbit are of the form $(\bar{p}ip, \bar{p}ip)$, and hence this orbit is isomorphic to $\mathbb{C}P^1$. We denote this orbit of diagonal elements as $\mathbb{C}P^1_{diag}$. Similarly, for $\phi = \frac{\pi}{2}$, all elements in the orbit are of the form $(-\bar{p}ip, \bar{p}ip)$, and hence we denote this orbit of anti-diagonal elements as $\mathbb{C}P^1_{adiag}$. In general the $SU(2)$-orbit is isomorphic to $SU(2)/\mathbb{Z}_2$, because $(\bar{p}ie^{2j\phi}p, \bar{p}ip)$ is invariant under the action $p \mapsto -p$. In summary, we have the following representations and orbits:

| | $\mathbb{C}P^1_{diag}$ | | $\mathbb{C}P^1_{adiag}$ |
|---|---|---|---|
| $\phi$ | $0$ | $\phi$ | $\pi/2$ |
| Unique rep. | $(i, i)$ | $(ie^{j\phi}, i)$ | $(k, i)$ |
| Orbit | $\frac{SU(2)}{\langle e^{i\vartheta} \rangle}$ | $\frac{SU(2)}{\langle \pm 1 \rangle}$ | $\frac{SU(2)}{\langle e^{i\vartheta} \rangle}$ |

Schroers & Singer (2021) identified a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action that is generated by two maps. First they consider the switching map $s$, that interchanges the values of the $\mathbb{C}P^1$'s inside $\mathbb{C}P^1 \times \mathbb{C}P^1$. Secondly, they consider the map $a$ that acts by the antipodal map on each component simultaneously. They denote the composition of $s$ and $a$ by $r$.

We claim we can identify the maps $s$, $a$ and $r$ by left multiplication of some unit quaternion. Indeed, for the Hopf fibration $p \mapsto \bar{p}ip$, the action by $p \mapsto j\,p$ descends to the antipodal map on the base space. Therefore, we identify the map $a$ with $p \mapsto j\,p$. Similarly, the $SU(2)$ action $p \mapsto ie^{j\phi}p$ transforms an element $(\bar{p}ie^{2j\phi}p, \bar{p}ip)$ into $(\bar{p}ip, \bar{p}ie^{2j\phi}p)$, which is the same as the switching map $s$. The composition $r$ of these maps must be given by $p \mapsto -ke^{j\phi}p$. Notice that the triple $\{-ke^{j\phi}, j, ie^{j\phi}\}$ is a rotation of the standard $\{i, j, k\}$ triple and so define

$$\hat{\imath} = -ke^{j\phi}, \;\; \hat{\jmath} = j, \;\text{ and }\; \hat{k} = ie^{j\phi}.$$

In summary, we have:

| | $\mathbb{C}P^1_{diag}$ | | $\mathbb{C}P^1_{adiag}$ |
|---|---|---|---|
| $\phi$ | $0$ | $\phi$ | $\pi/2$ |
| Orbit | $\frac{SU(2)}{\langle e^{\hat{k}\vartheta}\rangle}$ | $\frac{SU(2)}{\langle \pm 1\rangle}$ | $\frac{SU(2)}{\langle e^{\hat{\imath}\vartheta}\rangle}$ |
| $s$-action | $p \mapsto \hat{k}p$ | $p \mapsto \hat{k}p$ | $p \mapsto \hat{k}p$ |
| $r$-action | $p \mapsto \hat{\imath}p$ | $p \mapsto \hat{\imath}p$ | $p \mapsto \hat{\imath}p$ |
| $a$-action | $p \mapsto \hat{\jmath}p$ | $p \mapsto \hat{\jmath}p$ | $p \mapsto \hat{\jmath}p$ |

It follows that $\hat{k}$ acts trivially on $\mathbb{C}P^1_{diag}$ and as the antipodal map on $\mathbb{C}P^1_{adiag}$. Similarly, $\hat{\imath}$ acts trivially on $\mathbb{C}P^1_{adiag}$ and as the antipodal map on $\mathbb{C}P^1_{diag}$. The group $\langle \hat{\imath}, \hat{\jmath}, \hat{k}\rangle$ is the dihedral group and acts as $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $SU(2)/\langle \pm 1\rangle$.

In the paper of Schroers & Singer (2021), they consider the complement of $\mathbb{C}P^1_{adiag}$, which they denote as $\widetilde{AH}$. The Atiyah-Hitchin manifold is the quotient of $\widetilde{AH}$ by the group $\langle \hat{\imath}, \hat{\jmath}, \hat{k}\rangle$. In terms of the quaternionic coordinates, the orbits of these quotient spaces are given as:

| | $\mathbb{C}P^1_{diag}$ | | $\mathbb{C}P^1_{adiag}$ |
|---|---|---|---|
| $\widetilde{AH}$ | $\frac{SU(2)}{\langle e^{\hat{k}\vartheta}\rangle}$ | $\frac{SU(2)}{\langle \pm 1\rangle}$ | $\frac{SU(2)}{\langle e^{\hat{\imath}\vartheta}\rangle}$ |
| $\widetilde{AH}/\langle \hat{k}\rangle$ | $\frac{SU(2)}{\langle e^{\hat{k}\vartheta}\rangle}$ | $\frac{SU(2)}{\langle \hat{k}\rangle}$ | $\frac{SU(2)}{\langle e^{\hat{\imath}\vartheta}, \hat{\jmath}\rangle}$ |
| $\widetilde{AH}/\langle \hat{\imath}\rangle$ | $\frac{SU(2)}{\langle e^{\hat{k}\vartheta}, \hat{\jmath}\rangle}$ | $\frac{SU(2)}{\langle \hat{\imath}\rangle}$ | $\frac{SU(2)}{\langle e^{\hat{\imath}\vartheta}\rangle}$ |
| $\widetilde{AH}/\langle \hat{\imath}, \hat{k}\rangle$ | $\frac{SU(2)}{\langle e^{\hat{k}\vartheta}, \hat{\jmath}\rangle}$ | $\frac{SU(2)}{\langle \hat{\imath}, \hat{\jmath}, \hat{k}\rangle}$ | $\frac{SU(2)}{\langle e^{\hat{\imath}\vartheta}, \hat{\jmath}\rangle}$ |

The space $\widetilde{AH}$ and its quotients retract to $\mathbb{C}P^1_{diag}$ and its quotients. Therefore the Atiyah-Hitchin manifold is homotopic to $\frac{SU(2)}{\langle e^{\hat{k}\vartheta}, \hat{\jmath}\rangle}$. This is isomorphic to $\mathbb{R}P^2$, because the Hopf projection implies $\frac{SU(2)}{\langle e^{\hat{k}\vartheta}\rangle} \simeq S^2$ and $\hat{\jmath}$ descends to the antipodal map. The fundamental group of the Atiyah-Hitchin manifold is $\mathbb{Z}_2$ and is generated by the antipodal map. To find its double cover, we need to find a simply connected space and a free $\mathbb{Z}_2$ action such that their quotient is $\widetilde{AH}/\langle \hat{\imath}, \hat{\jmath}, \hat{k}\rangle$. This is satisfied for $\widetilde{AH}/\langle \hat{k}\rangle$ and the $\hat{\imath}$ action.

In order to compare the asymptotic Atiyah-Hitchin metric with the Taub-NUT metric, we have to consider the branched cover $\tilde{AH}/\langle \hat{\imath}\rangle$ instead. On here, a generic fiber is of the form $\frac{SU(2)}{\langle \hat{\imath} \rangle}$, which is a circle bundle over $S^2$ of degree $\pm 4$. In these coordinates, the circle action can be seen by left multiplication of $e^{\hat{\imath}\phi}$ on $\frac{SU(2)}{\langle \hat{\imath} \rangle}$. On the branched cover, the $\hat{k}$-action descends to the antipodal map on $\frac{SU(2)}{\langle e^{\hat{\imath}\phi} \rangle}$ and anti-commutes with the fiber, i.e. $\hat{k}e^{\hat{\imath}\theta} \cdot p = e^{-\hat{\imath}\theta}\hat{k} \cdot p$.

While in the Taub-NUT case the circle fiber is contractible, the circle fiber in the Atiyah-Hitchin manifold maps to the generator of $\mathbb{R}P^2$. This can be seen explicitly in the above picture: A generic fiber inside the Atiyah-Hitchin manifold can be parametrised by

$$e^{i\,t} \cdot (\bar{p}\hat{k}e^{\hat{\jmath}\phi}p, \bar{p}\hat{k}e^{-\hat{\jmath}\phi}p) \mapsto (\bar{p}e^{-\hat{\imath}\frac{t}{4}}\hat{k}e^{\hat{\jmath}\phi}e^{\hat{\imath}\frac{t}{4}}p, \bar{p}e^{-\hat{\imath}\frac{t}{4}}\hat{k}e^{-\hat{\jmath}\phi}e^{\hat{\imath}\frac{t}{4}}p).$$

When we retract it to the core ($\phi = 0$), the right hand side of the expression simplifies to $(\bar{p}\hat{k}e^{\hat{\imath}\frac{t}{2}}p, \bar{p}\hat{k}e^{\hat{\imath}\frac{t}{2}}p)$. This implies that a rotation along the fiber retracts to the map $(\bar{p}\hat{k}p, \bar{p}\hat{k}p) \to (-\bar{p}\hat{k}p, -\bar{p}\hat{k}p)$ inside $\mathbb{C}P^1_{diag}$. This is the antipodal map on $S^2$ which is the generator of $\pi_1(\mathbb{R}P^2)$.

*Remark* 2.7. There is a construction for ALF-$D_k$ gravitational instantons which gives an alternative construction for the Atiyah-Hitchin manifold. According to Hitchin et al. (1987) all hyperkähler structures $(I, J, K)$ on a $4n$-dimensional manifold $M$ can be uniquely encoded into complex geometric structures on the twistor space $Z := M \times S^2$. According to Chen & Chen (2019) all ALF-$D_k$ gravitational instantons can be constructed using this twistor method. This method was conjectured by Ivanov & Roček (1996) using a generalized Legendre transform developed by Lindström & Roček (1988). Cherkis & Kapustin (1999) confirmed this conjecture and Cherkis & Hitchin (2005) computed the metric more explicitly. Therefore, this metric is called the Cherkis–Hitchin–Ivanov–Kapustin–Lindström–Roček metric. One can ask whether the other ALF gravitational instantons have a gauge theoretic description. Cherkis & Kapustin (1999) claim that all ALF gravitational instantons can be seen as the moduli space of $U(2)$-monopoles of degree 1 or 2 with $k$ Dirac singularities.

### 'Periodic' gravitational instantons

Let us revisit the construction of the Gibbons-Hawking metric and see whether we can generalise it: Given a principal circle bundle $P$ over some $U \subseteq \mathbb{R}^3$, a connection $\eta$ and a positive harmonic function $h\colon U \to \mathbb{R}$, we can construct the Gibbons-Hawking metric

$$g^{GH} = h\, g_U + h^{-1}\eta^2.$$

In the proof of Proposition 2.3, we have seen that the orthonormal 2-forms

$$\omega_i^{GH} = \mathrm{d}\, x_i \wedge \eta + h\, \mathrm{d}\, x_j \wedge \mathrm{d}\, x_k$$

induce almost complex structures

$$I_i^* \,\mathrm{d}\, x_j = \begin{cases} \frac{-1}{h}\eta & i = j \\ -\operatorname{sgn}(ijk)\,\mathrm{d}\, x_k & i \neq j \end{cases}$$

which are integrable if and only if $\mathrm{d}\,\omega_i = 0$. This only happens when the Bogomolny equation $*\,\mathrm{d}\, h = \mathrm{d}\,\eta$ is satisfied.

To generalise this method for other base spaces, we assume that $B$ is some Riemannian 3-manifold and $h\colon B \to \mathbb{R}$ is a positive harmonic function with $[*\,\mathrm{d}\, h] \in H^2(B, \mathbb{Z})$. As before, we can retrieve a circle bundle $P \to B$ with a connection $\eta$ satisfying the Bogomolny equation and we can equip $P$ with the Gibbons-Hawking metric. To construct the Kähler forms however, we need to generalise the forms $\mathrm{d}\, x_1, \mathrm{d}\, x_2, \mathrm{d}\, x_3 \in \Omega^1(\mathbb{R}^3)$. We assume $B$ has three nowhere-vanishing 1-forms $\sigma_i$. The quaternion relations force $\sigma_i$ to be orthonormal and the almost complex structures are integrable if the $\sigma_i$'s are closed. In summary, the Gibbons-Hawking ansatz can be done over any flat, 3-dimensional, Riemannian manifold that is equipped with an orthonormal basis of nowhere-vanishing, closed 1-forms.

One example is the metric by Ooguri & Vafa (1996). It can be constructed by the Gibbons-Hawking ansatz on $B = (\mathbb{R}^2 \setminus \{0\}) \times S^1$, where the harmonic function is the Green's function. On the universal cover of $B$, this Green's function can be recovered by considering

$$h(x) = \mathrm{const} + \sum_{n \in \mathbb{Z}} \frac{1}{2|x - n \cdot \vec{z}|}.$$

This infinite sum does not converge and hence one has to renormalise it to make it well-defined. This has the consequence that $h$ cannot be positive everywhere. That is, near the singularity $h(x)$ approximates $\mathrm{const} + \frac{1}{2|x|}$ while near infinity $h(x) \simeq -\frac{1}{4\pi}\log(x_1^2 + x_2^2)$, where $x_1$ and $x_2$ are the coordinates on $\mathbb{R}^2$ inside $\mathbb{R}^2 \times S^1$. Therefore, the Gibbons-Hawking metric is positive definite near the singularity, but negative definite near infinity. For more details, see Gross & Wilson (2000).

Although the Ooguri-Vafa metric is not complete, 'periodic' versions of the Gibbons-Hawking ansatz are still useful in understanding the classification of gravitational instantons. Namely, Sun & Zhang (2021) have classified all gravitational instantons in terms of their asymptotic geometry. We have already seen that the asymptotic geometry of ALF gravitational instantons is described in terms of the Gibbons Hawking ansatz. A 'negative version' of the Ooguri-Vafa metric is used to describe the ALG* metric, which we explain in the following definition:

**Definition 2.8.** *A gravitational instanton $(M, g)$ is of type ALG* if there are $\epsilon, c > 0$, $k \in \mathbb{N}_{>0}$ and a diffeomorphism $\varphi$ between the asymptotic regions of $M$ and a $\mathbb{Z}_2$ quotient of the Gibbons-Hawking space for $h(x) = c + k \cdot \log(x_1^2 + x_2^2)$ on $(\mathbb{R}^2 \setminus \{0\}) \times S^1$ such that*

$$\|\nabla^j(\varphi_* g - g^{GH})\|_{g^{GH}} = \mathcal{O}(r^{-j-\epsilon})$$

*for all $j \in \mathbb{N}$ and this $\mathbb{Z}_2$ quotient is a lift of the action induced by the antipodal map on $\mathbb{R}^3$.*

**Definition 2.9.** *A gravitational instanton $(M, g)$ is of type ALG if there is a $k \in \{2, 3, 4, 6\}$, an $\epsilon > 0$, and a diffeomorphism $\varphi$ between the asymptotic regions of $M$ and a flat orbifold $(\mathbb{R}^2 \times T^2)/\mathbb{Z}_k$ such that*

$$\|\nabla^j(\varphi_* g - g^{flat})\|_{g^{flat}} = \mathcal{O}(r^{-j-\epsilon})$$

*for all $j \in \mathbb{N}$.*

ALG and ALG* gravitational instantons both have quadratic volume growth. However, for the ALG* case, the circle radius of the fiber (and hence the injectivity radius) decays with order $\frac{1}{\sqrt{\log(r)}}$, while for the ALG case, they have a lower bound. Also, the curvature decay for an ALG* gravitational instantons is $\frac{1}{r^2 \log r}$, while for ALG gravitational instantons it is at least $\frac{1}{r^{5/2}}$.

According to Chen & Chen (2021b) and Chen & Viaclovsky (2021) any ALG/ALG* gravitational instanton can be compactified to a rational elliptic surface by adding a singular elliptic fiber. If this fiber is $I_\nu^*$ for $\nu \in \{1,2,3,4\}$, the gravitational instanton is of type ALG*. If the fiber is $I_0^*$, $II$, $II^*$, $III$, $III^*$, $IV$ or $IV^*$, it is of type ALG. Moreover, Chen et al. (2021) found a Torelli theorem[3] for ALG and ALG* gravitational instantons.

### 'Doubly periodic' gravitational instantons

For the the Ooguri-Vafa metric we considered the Gibbons-Hawking ansatz on the quotient of $\mathbb{R}^3$ by a rank one lattice. Next we study the case of a rank two lattice. Just as the Ooguri-Vafa case, the Green's function on $\mathbb{R} \times T^2$ cannot be chosen positive globally[4], and so the Gibbons-Hawking ansatz yields a non-complete hyperkähler manifold. However, these examples are still used in the classification of gravitational instantons:

**Definition 2.10.** *A gravitational instanton $(M, g)$ is of type ALH if there are $\epsilon, c > 0$ and a diffeomorphism $\varphi$ between the asymptotic regions of $M$ and the Gibbons-Hawking space for $h(x) = c$ on $\mathbb{R}^+ \times T^2$ such that*

$$\|\nabla^j(\varphi_* g - g^{GH})\|_{g^{GH}} = \mathcal{O}(r^{-j-\epsilon})$$

*for all $j \in \mathbb{N}$.*

**Definition 2.11.** *A gravitational instanton $(M, g)$ is of type ALH* if there are $\epsilon, c > 0$, $k \in \mathbb{N}_{>0}$ and a diffeomorphism $\varphi$ between the asymptotic regions of $M$ and the Gibbons-Hawking space for $h(x) = c + k \cdot |x_1|$ on $\mathbb{R}^+ \times T^2$ such that*

$$\|\nabla^j(\varphi_* g - g^{GH})\|_{g^{GH}} = \mathcal{O}(r^{-j-\epsilon})$$

*for all $j \in \mathbb{N}$.*

ALH gravitational instantons have linear volume growth and their injectivity radius is bounded below. ALH* gravitational instantons however, have volume growth $r^{4/3}$

[3] See Remark 2.5.

[4] In Lemma 3.3 we will prove this explicitly.

and their injectivity radius decays with order $r^{-1/3}$. To see this, one applies the coordinate transform $s = x_1^{3/2}$ on the asymptotic Gibbons-Hawking metric and gets

$$g^{GH} \simeq \mathrm{d}\, s^2 + s^{2/3} \cdot g_{T^2} + s^{-2/3}\eta^2.$$

The curvature decay for ALH* is $r^{-2}$, while ALH has faster than quadratic curvature decay.

According to Chen & Chen (2021b) and T. C. Collins et al. (2020) any ALH/ALH* gravitational instanton can be compactified to a rational elliptic surface by adding a $I_k$-fiber. If this fiber is regular, i.e. $k = 0$, the gravitational instanton is of type ALH. If the elliptic fiber is not regular, the gravitational instanton is of type ALH*. Sun & Zhang (2021) have shown that $k$ must be between 1 and 9. A Torelli theorem for ALH gravitational instantons was found by Chen & Chen (2021b). T. Collins et al. (2022) and Lee & Lin (2022) determined the Torelli theorem for ALH* gravitational instantons.

There are two main examples of '(doubly) periodic' gravitational instantons. The first example is due to Tian & Yau (1990). They constructed gravitational instantons on the complement of a smooth anti-canonical divisor inside a weak del Pezzo surface. Just as the Calabi ansatz[5] yields a hyperkähler metric on an ample line bundle over a compact Calabi-Yau manifold, Tian and Yau constructed an approximate solution on the tubular neighbourhood of the divisor. They extended the Kähler form $\omega$ and a holomorphic volume form $\Omega$ globally and considered perturbations of the form $\omega_{TY} = \omega + i\partial\bar{\partial}\phi$. They found $\phi$ by solving the Monge-Ampère equation $(\omega + i\partial\bar{\partial}\phi)^2 = \frac{1}{2}\Omega \wedge \bar{\Omega}$. The function $\phi$ has exponential decay and hence the Tian-Yau metric has the same asymptotic geometry as their approximate solution, which is ALH*.

Continuing on this work, Hein (2010) constructed gravitational instantons on the complement of a (singular) elliptic fiber inside a rational elliptic surface. Using the classification by Kodaira (1963), Hein retrieved an explicit description of the neighbourhood of the (singular) fiber and he equipped it with a metric that is flat on the fibers. By extending these structures globally, he made an almost hyperkähler manifold and he perturbed it into a gravitational instanton by solving the Monge-

[5]See Calabi (1979). A short exposition in English can be found in Hein et al. (2022).

Ampère equations. Depending on the type of elliptic fibration, Hein constructed ALG-, ALG*-, ALH- or ALH*-type gravitational instantons.

## 2.2 Gluing constructions

In this thesis, we will give a gluing construction for gravitational instantons of type ALF, ALG, ALG*, ALH and ALH*. The method originates from Donaldson (2006) and is used often in the literature[6]. The idea is that we start with a non-complete, non-compact hyperkähler manifold with explicit models near the boundary. (See Figure 1.) We refer to this space as the bulk space.

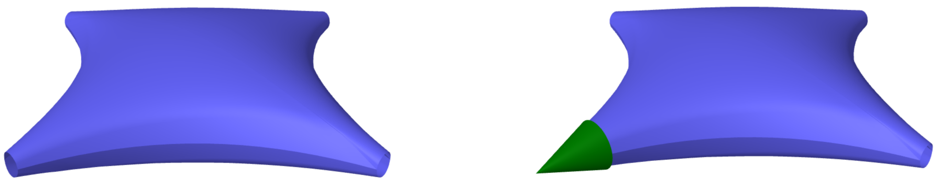


Figure 1: *Pictorial representation of our bulk space. In our case, we construct the bulk space using the Gibbons-Hawking ansatz. The metric around the singularities must approximate the Taub-NUT space (with mass $-4$), which is depicted as a green cone.*

In our case we will construct the bulk space using the Gibbons-Hawking ansatz. Just as the Ooguri-Vafa metric, we will consider $\mathbb{R}^3$ modulo a lattice of rank less than three and we remove a certain number of points where we will perform the gluing. The antipodal map on $\mathbb{R}^3$ induces a $\mathbb{Z}_2$ action on our base space and we assert that our setup is invariant under this involution. We also assert that the fixed points are part of the set of points we remove. Next, we pick the harmonic function such that, near the singularities, the Gibbons-Hawking space is modelled on the Taub-NUT space. More precisely, if a singularity is at a fixed point of the $\mathbb{Z}_2$ action, we model it on the Taub-NUT space with negative mass[7] $-4$ and otherwise we model it on the standard[8] Taub-NUT space. We control the model at infinity by our the choice of lattice and by the number of singularities we allow to form. Our bulk space will be the $\mathbb{Z}_2$ quotient of this Gibbons-Hawking space. More details will be given in

[6] See, e.g. Foscolo (2019), Fine et al. (2017), Hein et al. (2022) or Schroers & Singer (2021)

[7] That is, the Gibbons-Hawking ansatz on $\mathbb{R}^3 \setminus 0$ with $h = c + \frac{-4}{2|x|}$.

[8] That is, the Gibbons-Hawking ansatz on $\mathbb{R}^3 \setminus 0$ with $h = c + \frac{1}{2|x|}$.

Section 3.1.

In the second step of the gluing construction, we choose for each singularity a complete hyperkähler manifold that asymptotically shares the same topology and metric as the model space (See Figure 2). We refer to these complete manifolds as bubbles. In our case these bubbles will be rescaled versions of Taub-NUT spaces and Atiyah-Hitchin manifolds. Because the asymptotic geometry of these bubbles coincides with the geometry near the singularities, we can identify these regions and create a complete differentiable manifold. This will be the underlying manifold of our gravitational instanton. Using a partition of unity we equip this manifold with a metric that is approximately hyperkähler.

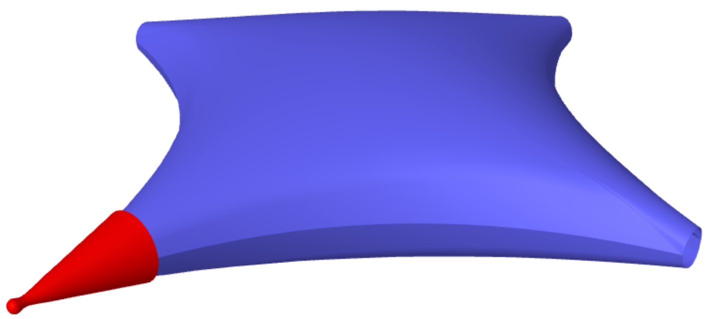


Figure 2: *Pictorial representation of the gluing construction. We complete the bulk space by adding 'bubbles' using a connected sum construction. In our case these bubbles will be rescaled versions of Taub-NUT spaces and Atiyah-Hitchin manifolds.*

Instead of constructing one gravitational instanton, we do this gluing procedure for a 1-parameter family of bulk spaces and bubbles. This extra parameter $\epsilon$, which we call the collapsing parameter, will measure the quality of our first approximation. We set up the gluing such that, in the limit, the error of our approximation vanishes. The explicit choice of our collapsing parameter will be explained in Section 3.3 and it relates to the size of the circle fiber. Therefore, in the limit where $\epsilon$ is zero, the manifold will not be a gravitational instanton, but will collapse to a flat 3-dimensional space. To solve this issue, we set $\epsilon$ sufficiently small and use a separate perturbation argument to turn the space into a genuine hyperkähler manifold.

### The elliptic problem

To perturb the approximate solution, we phrase the hyperkähler conditions as an elliptic PDE which we will solve using the inverse function theorem. Before, we defined a hyperkähler manifold as a Riemannian manifold $(M, g)$ with three compatible Kähler structures that satisfy the quaternion relations. Although this definition is beautiful in its simplicity, it is hard to solve. Therefore, we will introduce an alternative formalism in terms of Kähler forms which will be used to set up the deformation problem. The perturbation argument explained here is a slightly modified version of that used in Schroers & Singer (2021).

Given a hyperkähler manifold $(M, g, I_1, I_2, I_3)$, consider its Kähler forms $\omega_i$. Instead of treating all Kähler forms equally, we fix a complex structure $I_1$ and consider the holomorphic volume form $\omega_{\mathbb{C}} = \omega_2 + \sqrt{-1}\omega_3$. Because Kähler forms are elements of $\Omega^{1,1}(M)$ and holomorphic volume forms are elements of $\Omega^{2,0}(M)$, $\omega_1 \wedge \omega_{\mathbb{C}} = 0$. By varying the complex structures $I_i$, we find

$$\omega_i \wedge \omega_j = 0 \text{ for all } i \neq j.$$

This implies that all three Kähler forms are orthogonal with respect to $g$. At the same time, the hyperkähler condition requires the three complex structures to be Kähler with respect the same Riemannian metric $g$. This implies $\frac{1}{2}\omega_i \wedge \omega_i = \mathrm{Vol}^g$ for all $i \in \{1, 2, 3\}$ and hence $\omega_i$ is a pointwise orthonormal frame on $\Lambda^+T^*M$. In general, the triple we get from the gluing construction is not orthonormal. However, for our analysis it will be sufficient if this triple is linearly independent.

**Definition 2.12.** *Let $M$ be a four dimensional manifold and let $\omega = (\omega_1, \omega_2, \omega_3)$ be a triple of 2-forms.*

1. *We say $\omega$ is a definite triple if there is a volume form $\mu$ and a positive-definite matrix $P \in C^\infty(M) \otimes \mathrm{Sym}^2(\mathbb{R}^3)$ such that*

$$\frac{1}{2}\omega_i \wedge \omega_j = P_{ij} \cdot \mu.$$

2. *We say $\omega$ is an orthonormal triple if it is a definite triple where $P$ is the identity matrix and*

$$\mu = \frac{1}{6}\sum_k \omega_k \wedge \omega_k.$$

According to Donaldson (2006) there is a unique conformal class of metrics for any definite triple such that $\omega_i$ spans the space of self-dual forms. This metric $g$ can be chosen uniquely if we also fix the volume form. The choice of volume form we made in Definition 2.12 is convenient, because it makes the Kähler forms from a hyperkähler metric automatically orthonormal.

**Definition 2.13.** *Let $M$ be a 4-dimensional manifold and let $\omega$ be a triple of 2-forms. We say that $\omega$ is a hyperkähler triple if*

1. *it is a triple of closed 2-forms, and*
2. *it is an orthonormal triple.*

Given a hyperkähler manifold, the induced triple of Kähler forms must be a hyperkähler triple. According to Donaldson (2006), the converse is also true, i.e. a hyperkähler triple induces a unique hyperkähler metric. Hence, the use of hyperkähler triples gives us an alternative definition for hyperkähler manifolds which is more algebraic. From now on we study hyperkähler triples instead of hyperkähler metrics.

We return to the gluing construction and we assume we found a definite triple of closed 2-forms $\omega$. Assume there exists a triple of 1-forms $a$ such that

$$\tilde{\omega} = \omega + \mathrm{d}\, a$$

is a hyperkähler triple. We will solve this for $a$. Because $\tilde{\omega}$ is hyperkähler, the expression $\tilde{\omega}_i \wedge \tilde{\omega}_j - \frac{1}{3}\delta_{ij}\sum_k \tilde{\omega}_k \wedge \tilde{\omega}_k$ is a traceless, symmetric $3 \times 3$ matrix with values in $\Omega^4(M)$. Therefore, we consider the projection map

$$\begin{aligned} \mathrm{Tf}\colon\ \mathrm{Mat}_{3\times 3}(\mathbb{R}) \otimes \Omega^4(M) &\to \mathrm{Sym}^2_0(\mathbb{R}^3) \otimes \Omega^4(M) \\ P \otimes \mu &\mapsto \left(\frac{1}{2}P + \frac{1}{2}P^* - \frac{1}{3}\,\mathrm{Tr}(P)\,\mathrm{Id}\right) \otimes \mu, \end{aligned} \tag{1}$$

and our goal is to find $a \in \Omega^1(M) \otimes \mathbb{R}^3$ such that

$$\mathrm{Tf}((\omega + \mathrm{d}\, a)^2) = \mathrm{Tf}(\omega \wedge \omega) + 2\,\mathrm{Tf}(\mathrm{d}\, a \wedge \omega) + \mathrm{Tf}(\mathrm{d}\, a \wedge \mathrm{d}\, a) = 0. \tag{2}$$

This does not have a unique solution, because $\Omega^1(M)\otimes\mathbb{R}^3$ has rank 12, but $\mathrm{Sym}^2_0(\mathbb{R}^3)\otimes$

$\Omega^4(M)$ is only a rank 5 vector bundle. In order to solve this issue, we first remove the gauge freedom $a \mapsto a + \mathrm{d}\, f$: According to Donaldson (2006), there is a unique metric $g$ such that $\omega_i$ span $\Omega^+(M)$ and $\mathrm{Vol}^g = \frac{1}{3}\sum_k \omega_k \wedge \omega_k$. We fix the gauge by assuming $\mathrm{d}^*\, a = 0$. In order to fix all remaining 9 degrees of freedom, we also assume that $a$ satisfies

$$\mathrm{d}\, a \wedge \omega = \mathrm{d}^+\, a \wedge \omega = -\frac{1}{2}\,\mathrm{Tf}\,(\omega \wedge \omega + \mathrm{d}\, a \wedge \mathrm{d}\, a)\,. \tag{3}$$

Next, recall that $\omega_i$ span $\Omega^+(M)$ and the wedge product is a non-degenerate pairing on $\Omega^+$. Therefore, the map

$$\begin{aligned} \Lambda\colon \Omega^+(M) \otimes \mathbb{R}^3 &\to \mathrm{Mat}_{3\times 3}(\mathbb{R}) \otimes \Omega^4(M) \\ \sigma &\mapsto \sigma \wedge \omega \end{aligned} \tag{4}$$

is a bijection and Equation 3 is equivalent to

$$\mathrm{d}^+\, a = -\frac{1}{2}\Lambda^{-1}\,\mathrm{Tf}(\omega \wedge \omega + \mathrm{d}\, a \wedge \mathrm{d}\, a). \tag{5}$$

Combining Equation 3 with the gauge fix $\mathrm{d}^*\, a = 0$, we conclude $a$ must satisfy

$$(\mathrm{d}^* + \mathrm{d}^+)\, a = -\frac{1}{2}\Lambda^{-1}\,\mathrm{Tf}(\omega \wedge \omega + \mathrm{d}\, a \wedge \mathrm{d}\, a).$$

Our choice of gauge is convenient, because $(\mathrm{d}^* + \mathrm{d}^+)$ is a Dirac operator:

**Lemma 2.14.** *The operator*

$$\begin{aligned} \not{D}\colon \Omega^0(M) \oplus \Omega^1(M) \oplus \Omega^\mp(M) &\to \Omega^0(M) \oplus \Omega^1(M) \oplus \Omega^\pm(M) \\ f &\mapsto \mathrm{d}\, f & f \in \Omega^0(M) \\ a &\mapsto (\mathrm{d}^* + \mathrm{d}^\pm)a & a \in \Omega^1(M) \\ \sigma &\mapsto 2\,\mathrm{d}^*\,\sigma & \sigma \in \Omega^\mp(M) \end{aligned}$$

*is a Dirac operator such that $\not{D}^{\,2}$ equals the Hodge Laplacian.*

*Proof.* Consider the canonical Clifford structure with the Dirac operator $\mathrm{d} + \mathrm{d}^*$ on the space of forms (See Example 3.19 in Roe (1998)). The volume form and the Clifford action induce a $\mathbb{Z}_2$ grading on $\Lambda^\bullet T^*M$ and decompose it as $S^+ \oplus S^-$. Each of these subbundles can be identified as

$$S^{\pm} = \Omega^0(M) \oplus \Omega^1(M) \oplus \Omega^{\pm}(M).$$

Under this identification $\mathrm{d} + \mathrm{d}^*$ becomes $\not{D}$ and hence $\not{D}$ is a Dirac operator. $\not{D}^{\,2}$ can be calculated explicitly, which will show $\not{D}^{\,2}$ is the Hodge Laplacian. $\square$

We assume that $a$ lies in the image of $\not{D}\colon (\Omega^0(M)\oplus\Omega^+(M))\otimes\mathbb{R}^3 \to \Omega^1(M)\otimes\mathbb{R}^3$. This has the advantage that the linearised version of Equation 5 is the Hodge Laplacian and that $a$ can be described by a section of a trivial bundle. Moreover, if we write $a = \not{D}\,(u+\zeta)$ with $u \in \Omega^0(M)\otimes\mathbb{R}^3$ and $\zeta \in \Omega^+(M)\otimes\mathbb{R}^3$, then $u$ and $\zeta$ must satisfy

$$\begin{aligned} \Delta\zeta &= -\frac{1}{2}\Lambda^{-1}\operatorname{Tf}(\omega\wedge\omega) - 2\Lambda^{-1}\operatorname{Tf}(\mathrm{d}\,\mathrm{d}^*\zeta\wedge\mathrm{d}\,\mathrm{d}^*\zeta), \qquad (6)\\ \Delta u &= 0. \end{aligned}$$

We fix the gauge $\mathrm{d}^* a = 0$ by setting $u = 0$. We will solve Equation 6 using the version of the inverse function theorem given in Lemma 6.15 in Foscolo (2019):

> **Theorem 2.15** (Inverse function theorem). *Let $F(x) = F(0) + L(x) + N(x)$ be a smooth function between Banach spaces such that there exist $r, q, C > 0$ satisfying*
>
> 1. *$L$ is an invertible linear operator with $\|L^{-1}\| < C$,*
> 2. *$\|N(x) - N(y)\| \le q\cdot\|x+y\|\cdot\|x-y\|$ for all $x, y \in B_r(0)$, and*
> 3. *$\|F(0)\| < \min\left\{\frac{1}{4qC^2}, \frac{r}{2C}\right\}$.*
>
> *Then, there exists a unique $x$ in the domain of $F$ such that $F(x) = 0$ and $\|x\| \le 2C\|F(0)\|$.*

Heuristically, condition 2 of Theorem 2.15 requires the non-linear part of $F$ to be 'quadratic'. (Compare this condition to the identity $(a+b)(a-b) = a^2 - b^2$.) A quick glance at Equation 6 suggests that this is indeed true. Similarly, condition 3 gives us a requirement for how good our approximate solution must be. We will use the collapsing parameter to satisfy this condition.

Lastly we need to find suitable Banach spaces such that the Hodge Laplacian on $\Omega^+(M)\otimes\mathbb{R}^3$ is invertible with bounded inverse. This is not obvious and proving this will be the main bulk of this thesis. However a small calculation will show it is sufficient to study the Laplacian acting on functions instead. To see this, trivialise $\zeta \in \Omega^+(M) \otimes \mathbb{R}^3$ into $\zeta_i = u_{ij}\,\omega_j$ and use Riemann normal coordinates $\{x_k\}$. By the Weitzenböck formula (Roe (1998) Equation 3.8),

$$\Delta(u_{ij}\omega_j) = \not{D}^{\,2}(u_{ij}\omega_j) = -\nabla^k\nabla_k(u_{ij}\omega_j) + \not{R}\,(u_{ij}\omega_j),$$

where $\not{R}$ is the Clifford contraction of the Riemann curvature tensor. Using the trivialisation of $\zeta$ and the fact that the Clifford contraction is $C^\infty$-linear,

$$\begin{aligned}\Delta(u_{ij}\omega_j) &= (\Delta u_{ij})\cdot\omega_j - 2\nabla^k u_{ij}\cdot\nabla_k\,\omega_j + u_{ij}\cdot\left[-\nabla^k\nabla_k\,\omega_j + \not{R}\,\omega_j\right]\\ &= (\Delta u_{ij})\,\omega_j - 2\nabla_{\nabla u_{ij}}\,\omega_j + u_{ij}\cdot\not{D}^{\,2}(\omega_j).\end{aligned}$$

The term $\not{D}^{\,2}(\omega_j)$ will vanish, because $\omega_j$ is closed and self-dual. When $\omega$ is a hyperkähler triple, $\nabla\omega = 0$ and hence $\Delta(u_{ij}\omega_j) = (\Delta u_{ij})\omega_j$. We expect that, when $\omega$ is sufficient close to being hyperkähler, the Hodge Laplacian on $\Omega^+(M)$ and the Laplacian on functions define equivalent operators.

# 3 Almost hyperkähler manifolds

In 1997, Ashoke Sen proposed a gluing construction for $D_k$-ALF instantons as an application in string theory. Namely, in his paper, Sen (1997) considered $n$ D6-branes with an orientifold plane in type IIA string theory (See Figure 3). In M-theory a D6-brane corresponds to the Cartesian product of $\mathbb{R}^{1,6}$ and a Taub-NUT space. Similarly an orientifold plane is related to the Cartesian product of $\mathbb{R}^{1,6}$ and the Atiyah-Hitchin manifold. Due to this product structure, Sen assumed that his 11-dimensional manifold decomposed into $\mathbb{R}^{1,6}$ and a 4-dimensional hyperkähler manifold $M$. He claims that $M$ can be constructed as follows: He started with the

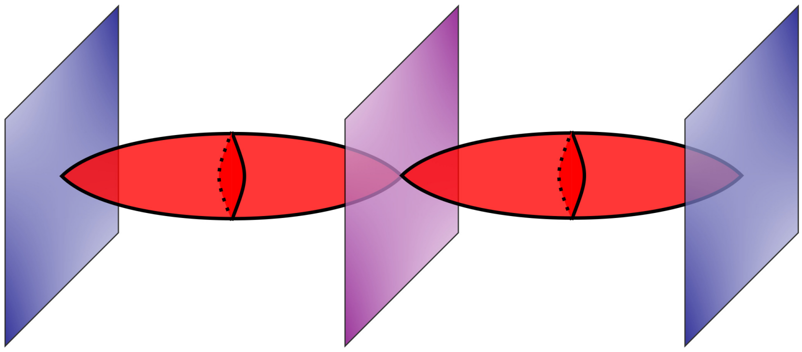



Figure 3: *Graphical depiction of two D6-branes and one orientifold in type IIA string theory.*

Gibbons-Hawking Ansatz on a punctured $\mathbb{R}^3$ with the harmonic function

$$h(x) = 1 + \frac{-4}{2|x|} + \sum_i \frac{1}{2|x+p_i|} + \frac{1}{2|x-p_i|},$$

where $p_i \neq 0$ are the positions of the D6-branes. Sen only considered the region of the Gibbons-Hawking space where $h$ is strictly positive, because near the origin the metric degenerates. The Gibbons-Hawking metric near the origin approximates

$$\left(\text{const} - \frac{2}{|x|} + \mathcal{O}(|x|^2)\right) g_{\mathbb{R}^3} + \frac{1}{\text{const} - \frac{2}{|x|} + \mathcal{O}(|x|^2)}\eta^2,$$

which approximates the Taub-NUT metric with mass $-4$. Similarly, this Gibbons-Hawking metric approximates the standard Taub-NUT metric near the singularities $p_i$.

Secondly, he considered the $\mathbb{Z}_2$ action induced by the antipodal map on $\mathbb{R}^3$. He lifted this involution such that near the origin it coincides with the projection map to the Atiyah-Hitchin manifold from its branched double cover. The metric is invariant under this involution and hence Sen quotiented the circle bundle by this $\mathbb{Z}_2$ action. He claimed that the region near $p_i$ approximates the Taub-NUT space. Similarly, he claimed that near the origin the metric approximates the Atiyah-Hitchin metric and he claimed that this can be made complete by gluing in these spaces. Although Sen never did this gluing explicitly, Schroers & Singer (2021) formalised his argument in their quest of finding geometric models of matter M. F. Atiyah et al. (2012).

In this chapter we extend the work of Sen and explain the construction of our approximate hyperkähler manifold. In Section 3.1 we construct the bulk space using the Gibbons-Hawking ansatz. We explain our choices for the base space and harmonic function and by studying the Green's functions we will find explicit estimates. After we find the circle bundle and involution, we do a parameter count for our choice of connection in Section 3.2. Next, we return to the construction of the bulk space and introduce the collapsing parameter in Section 3.3 and finish the construction in Section 3.4. Before we end the chapter, we study the topology and compare it to known classifications in Section 3.5.

## 3.1 The bulk space

Similarly to what Sen did, we will construct the bulk space using the Gibbons-Hawking ansatz. Instead of considering $\mathbb{R}^3$, we consider $\mathbb{R}^3$ modulo a non-maximal lattice $L$, i.e. a lattice whose rank is less than three. Before we apply the Gibbons-Hawking ansatz, we assign a certain number of points where singularities are allowed to form. After we apply the Gibbons-Hawking ansatz, we lift the antipodal map of $\mathbb{R}^3$ to the circle bundle and consider its $\mathbb{Z}_2$ quotient as our bulk space. These spaces are the main topic of this thesis and will be used throughout without reference. Hence, we will define them formally:

**Definition 3.1.** *Fix once and for all a non-maximal lattice $L$ on $\mathbb{R}^3$.*

- *We call the quotient $B := \mathbb{R}^3/L$, endowed with the flat metric, the base space.*
- *We refer to the map $\tau\colon B \to B$ that is induced from the map $x \mapsto -x$ on $\mathbb{R}^3$ as the antipodal map.*

- *We denote the fixed point set of $\tau$ by $\{q_j\}$ and we call $q_j$ a fixed point or a fixed point singularity. Unless specified otherwise, we call the action induced by $\tau$ on $B$, the $\mathbb{Z}_2$ action on $B$.*

Because $L$ is non-maximal, the basepace $B$ can only be diffeomorphic to $\mathbb{R}^3$, $\mathbb{R}^2 \times S^1$ or $\mathbb{R} \times T^2$. We will see later that each case yields different kinds of gravitational instantons and will require different kinds of analysis. When we distinguish these cases we will write $B = \mathbb{R}^3$, $B = \mathbb{R}^2 \times S^1$ or $B = \mathbb{R} \times T^2$.

When the lattice $L$ is trivial, the only fixed point is the origin. However, in the other cases we have two or four fixed points respectively. Just as in Sen (1997), we pick a certain number of points which we remove from $B$. It turns out we need to impose some restrictions on the number of singularities, which are given in the following definition. In Lemma 3.4 we see the necessity of these requirements.

**Definition 3.2.** *Fix once and for all a finite set of points $\{p_i\} \in (B \setminus \{q_j\})/\mathbb{Z}_2$. We call an element $p_i$ a non-fixed point or a non-fixed singularity. When $B \neq \mathbb{R}^3$, the number of non-fixed points must satisfy:*

| | $\mathbb{R}^2 \times S^1$ | $\mathbb{R} \times T^2$ |
|---|---|---|
| *Maximum number of non-fixed points* | 4 | 8 |

*Now let $\epsilon > 0$ be small and denote*

$$B' = B \setminus \cup_i \{\pm p_i\} \setminus \cup_j \overline{B}_{4\epsilon}(q_j). \tag{7}$$

*We call $B'$ the punctured base space.*

Our goal is to apply the Gibbons-Hawking ansatz over $B'$ that approximates the Taub-NUT metric near the $p_i$'s and the branched double cover of the Atiyah-Hitchin metric near the fixed points $q_j$. For this we first need to construct a suitable harmonic function with the correct asymptotics near the singularities. For the Taub-NUT metric this requires that the harmonic function must diverge as $\frac{1}{2r}$ at $\pm p_i$. For the Atiyah-Hitchin metric this requires that the harmonic function must diverge as $\frac{-2}{r}$ at $q_j$. Because the Gibbons-Hawking ansatz requires harmonic function to be positive, we had to remove a ball around the points $q_j$. This explains the definition of the punctured base space $B'$. The exact choice of radius for $B_{4\epsilon}(q_j)$ in $B'$ will be determined later when we study the gluing in more detail. For now it is sufficient that the radius is small enough such that the balls $B_{4\epsilon}(q_j)$ are pairwise disjoint and $B'$ is connected.

### Harmonic function

Recall that the Green's function on $\mathbb{R}^3$ is given by $G(x, x') = \frac{1}{4\pi|x-x'|}$. This implies that near $p_i$, the harmonic function $h$ must satisfy $\Delta h = 2\pi\delta(x - p_i)$. Similarly, near the fixed point singularities, the harmonic function $h$ must satisfy $\Delta h = -8\pi\delta(x - q_j)$. By linearity this can be done globally, and hence we need to solve

$$\Delta h = -8\pi \sum_j \delta(x - q_j) + 2\pi \sum_i \delta(x - p_i) + \delta(x + p_i)$$

on $B$. To solve this, we first recall the Green's function for $\mathbb{R}^3/L$.

**Lemma 3.3.** *Let $p \in B$. There exists a smooth function $G$ on $B \setminus \{p\}$ that solves*

$$\Delta G = 2\pi\delta(x - p) \text{ on } B,$$

*is invariant under the involution centred at $p$, and has the following asymptotic expansion near infinity:*

$$G(x, y, z) = \begin{cases} \frac{1}{2\sqrt{x^2+y^2+z^2}} + \mathcal{O}((x^2 + y^2 + z^2)^{-1}) & B = \mathbb{R}^3 \\ -\frac{1}{4\pi \operatorname{Vol}(S^1)} \log(x^2 + y^2) + \mathcal{O}(e^{-2\pi\sqrt{x^2+y^2}/\operatorname{Vol}(S^1)}) & B = \mathbb{R}^2 \times S^1 \\ -\frac{\pi}{\operatorname{Vol}(T^2)}|x| + \mathcal{O}(e^{-|x|}) & B = \mathbb{R} \times T^2. \end{cases}$$

*Proof.* On $\mathbb{R}^3$, the function $G$ can be found explicitly and is given by $G(x, y, z) = \frac{1}{2|x|}$. For the rest of this proof we work out the case $B = \mathbb{R} \times T^2$. The case $B = \mathbb{R}^2 \times S^1$ is similar and an alternative proof can be found in Gross & Wilson (2000), Lemma 3.1.

We equip $\mathbb{R} \times T^2$ with coordinates $(x, y, z)$ such that the metric is given by $g = \mathrm{d}\,x^2 + g_{T^2}$. Without loss of generality we assume that $p$ is at $(0, 0, 0)$. Let $N \in \mathbb{N}$ and consider the function

$$G^N(x, y, z) = \frac{\pi}{\operatorname{Vol}(T^2)} \left[ -|x| + \sum_{0<|(m,n)|<N} \frac{1}{\sqrt{m^2 + n^2}} e^{-\sqrt{m^2+n^2}|x|} e^{i(my+nz)} \right].$$

The function $G^N + \frac{\pi}{\mathrm{Vol}(T^2)}|x|$ is an $L^2$ function on $\mathbb{R} \times T^2$, because the summand is exponentially decaying. By construction $\overline{G^N} = G^N$, and so $G^N$ is real valued. We claim that $G^N + \frac{\pi}{\mathrm{Vol}(T^2)}|x|$ is a Cauchy sequence in $L^2$. Indeed, for any $M > N$,

$$\left\|G^M - G^N\right\|_{L^2(R\times T^2)} = \frac{\pi^2}{\mathrm{Vol}(T^2)} \sum_{N\leq|(m,n)|<M} \frac{1}{(m^2+n^2)^{3/2}} = \mathcal{O}(N^{-1}).$$

Define the limit of $G^N$ as $G$. We show that $G$ solves $\Delta G = 2\pi\delta$ as a distribution. For this, let $f \in C_c^\infty(\mathbb{R} \times T^2)$, $\delta > 0$ and $D = (\mathbb{R} \setminus B_\delta(0)) \times T^2$. We consider $\langle \Delta f, G\rangle_{L^2(D)}$ for some sufficiently small $\delta$. By integration by parts,

$$\begin{aligned}\langle \Delta f, G\rangle_{L^2(D)} &= \lim_{N\to\infty} \langle \Delta f, G^N\rangle_{L^2(D)} \\ &= \lim_{N\to\infty} \langle f, \Delta G^N\rangle_{L^2(D)} + \int_{S_\delta(0)\times T^2} \left(\overline{G^N}\frac{\partial f}{\partial x} - f\frac{\partial \overline{G^N}}{\partial x}\right) \mathrm{Vol}_T^2 .\end{aligned}$$

The function $G^N$ is defined as a sum of harmonic functions, and therefore $\langle f, \Delta G^N\rangle_{L^2(D)} = 0$. Next we consider the Fourier decompositions of $f$ and $G$, i.e. $f = \sum_{k,l\in\mathbb{Z}} \hat{f}_{kl}(x)e^{i(ky+lz)}$ and $G = \sum_{m,n\in\mathbb{Z}} \hat{G}_{mn}e^{i(my+nz)}$. This simplifies our calculation of $\langle \Delta f, G\rangle_{L^2(D)}$, as

$$\begin{aligned}\langle \Delta f, G\rangle_{L^2(D)} &= \lim_{N\to\infty} \sum_{\substack{k,l\in\mathbb{Z}\\ 0<|(m,n)|<N}} \int_{S_\delta(0)\times T^2} \left(\overline{\hat{G}_{mn}}\frac{\partial \hat{f}_{kl}}{\partial x} - \hat{f}_{kl}\frac{\partial \overline{\hat{G}_{mn}}}{\partial x} e^{i((k-m)y+(l-n)z)}\, \mathrm{Vol}_T^2\right) \\ &= \lim_{N\to\infty} \mathrm{Vol}(T^2) \sum_{|(m,n)|<N} \left[\overline{\hat{G}_{mn}}\frac{\partial \hat{f}_{mn}}{\partial x} - \hat{f}_{mn}\frac{\partial \overline{\hat{G}_{mn}}}{\partial x}\right]_{-\delta}^{\delta} .\end{aligned}$$

We evaluate the right hand side explicitly, which is

$$\begin{aligned}\langle \Delta f, G\rangle_{L^2(D)} =& -\delta\pi\left(\frac{\partial \hat{f}_{00}}{\partial x}(\delta) - \frac{\partial \hat{f}_{00}}{\partial x}(-\delta)\right) + \pi\left(\hat{f}_{00}(\delta) + \hat{f}_{00}(-\delta)\right) \\ &+ \pi \sum_{(m,n)\neq(0,0)} \frac{1}{\sqrt{m^2+n^2}} e^{-\sqrt{m^2+n^2}|\delta|} \left(\frac{\partial \hat{f}_{mn}}{\partial x}(\delta) - \frac{\partial \hat{f}_{mn}}{\partial x}(-\delta)\right) \\ &+ \pi \sum_{(m,n)\neq(0,0)} e^{-\sqrt{m^2+n^2}|\delta|} \left(+\hat{f}_{mn}(\delta) + \hat{f}_{mn}(-\delta)\right) .\end{aligned}$$

When we take the limit $\delta \to 0$, we conclude

$$\langle \Delta f, G \rangle_{L^2(\mathbb{R}\times T^2)} = 2\pi \hat{f}_{00}(0) + 2\pi \sum_{(m,n)\neq(0,0)} \hat{f}_{mn}(0) = 2\pi f(0,0,0).$$

Finally, we investigate whether $G$ is smooth outside $p$. For any open $U$ away from $p$, the function $G$ satisfies $\Delta G = 0$ as a distribution. According to Folland (1995) Proposition 6.33, $G$ is an element of $W^{k,2}(U)$ for all $k \in \mathbb{N}$. By the Sobolev inequality, $G$ must be smooth. □

Taking linear combinations of $G$ we are now able to construct the harmonic function that is required for the Gibbons-Hawking ansatz:

**Lemma 3.4.** *Write $\#\{p_i\}$ for the number of pairs $p_i$ in $(B - \{q_i\})/\mathbb{Z}_2$. Let $G$ be the Green's function defined in Lemma 3.3 and consider*

$$h = -4\sum_j G(x-q_j) + \sum_i \left(G(x-p_i) + G(x+p_i)\right).$$

*Let $r$ be the Euclidean distance from the origin on $\mathbb{R}^3$, $\mathbb{R}^2$ or $\mathbb{R}$ when $B = \mathbb{R}^3$, $B = \mathbb{R}^2 \times S^1$ or $B = \mathbb{R} \times T^2$ respectively.*

*(a) Near infinity,*

$$h = \begin{cases} \frac{2\cdot\#\{p_i\}-4}{2r} + \mathcal{O}(r^{-3}) & \text{if } B = \mathbb{R}^3 \\ \beta\cdot(8 - 2\cdot\#\{p_i\})\cdot\log(r) + \mathcal{O}(r^{-2}) & \text{if } B = \mathbb{R}^2\times S^1 \\ \beta\cdot(16 - 2\#\{p_i\})\cdot r + \mathcal{O}(e^{-r}) & \text{if } B = \mathbb{R}\times T^2 \end{cases}$$

*for some $\beta > 0$, and $\beta$ only depends on the lattice $L$.*

*(b) Near the fixed points $q_j$, $h(x) = \alpha_j - \frac{2}{|x-q_j|} + \mathcal{O}(|x-q_j|^2)$ for some $\alpha_j > 0$. Near the non-fixed points $\pm p_i$, $h(x) = \alpha_i + \frac{1}{2|x\mp p_i|} + \mathcal{O}(|x\mp p_i|^2)$ for some $\alpha_i > 0$.*

*(c) There exists an $\delta > 0$ such that $\epsilon^{-1} + h$ is a harmonic function on $B' = B \setminus \cup_i\{\pm p_i\} \setminus \cup_j \overline{B}_{4\epsilon}(q_j)$ which is greater than $\frac{1}{2}$ for all $0 < \epsilon < \delta$.*

*(d) The only maps that satisfy*

1. $\Delta\tilde{h} = -8\pi\sum_j \delta(x-q_j) + 2\pi\sum_i \delta(x-p_i) + \delta(x+p_i)$,

*2. $\tilde{h}$ is bounded below on $B'$,*

*are the maps $\tilde{h} = h + c$ for some constant $c \in \mathbb{R}$.*

*Proof.* Part (a): These estimates follow from the expansion given in Lemma 3.3. When $B = \mathbb{R}^3$ or $B = \mathbb{R}^2 \times S^1$, the leading error term in $\mathcal{O}(r^{-2})$ or $\mathcal{O}(r^{-1})$ respectively, disappears due to the $\mathbb{Z}_2$ invariance.

Part (b): Recall that on any small ball $U$ centred at $p_i$ in $B$, the function $h$ satisfies $\Delta h = 2\pi\delta(x - p_i)$. The function $\frac{1}{2|x-p_i|}$ also satisfies this equation. This implies that $h - \frac{1}{2|x-p_i|}$ is a harmonic function on $U$. Expanding this difference in terms of spherical harmonics yields

$$h(x) - \frac{1}{2|x - p_i|} = \alpha_j + \mathcal{O}(|x - p_i|^2)$$

for some $\alpha_i \in \mathbb{R}$. Similarly, we can expand $h$ near the fixed point $q_j$. In this case the linear term will vanish due to the $\mathbb{Z}_2$ invariance of $h$.

Part (c): We consider $\epsilon^{-1} + h$. We already know that it is harmonic on $B'$, and hence we only need to find when it is greater than $\frac{1}{2}$. By the maximum principle any harmonic function attains its minimum on the boundary, where we have explicit estimates. Near the point $p_i$, the function $h$ diverges to $+\infty$ with rate $\frac{1}{r}$. Hence, it is positive near this boundary. On the boundary near the fixed point $q_j$ we have the estimate

$$\epsilon^{-1} + h = \epsilon^{-1} + \alpha_j - \frac{1}{2}\epsilon^{-1} + \mathcal{O}(\epsilon^2)$$

This diverges to $+\infty$ as $\epsilon$ tends to 0. Lastly, we consider the boundary at infinity. At this boundary, the function $h$ can only attain 0 or $\pm\infty$, and using the conditions specified in Definition 3.2 the case $h|_\infty = -\infty$ is discarded. Hence $h$ is bounded below and $\epsilon^{-1} + h$ can be chosen strictly positive, greater than $\frac{1}{2}$.

Part (d): We only show uniqueness. Suppose that $\tilde{h}$ satisfies

1. $\Delta\tilde{h} = -8\pi \sum_j \delta(x - q_j) + 2\pi \sum_i \delta(x - p_i) + \delta(x + p_i)$, and
2. $\tilde{h}$ is bounded below on $B'$.

Then, $u := \tilde{h} - h$ is a harmonic function on $B$ which can be lifted to a harmonic function on $\mathbb{R}^3$. We claim that $u = \mathcal{O}(r)$. Indeed, due to part (a) the lower bound of $u$ diverges at most linearly to $-\infty$ and hence we only need to study the upper bound. For this, fix $r > 0$ sufficiently large and consider the map $u_r^+(x) := u(x)+1-\inf_{y\in B_{2r}(0)} u(y)$. This is strictly positive on $B_{2r}(0)$ and hence the Harnack inequality implies for all $x \in B_r(0)$,

$$u_r^+(x) \leq 6\, u_r^+(0).$$

For sufficiently large $r$, this can be rewritten as

$$u(x) \leq C + 5 \sup_{y\in B_{2r}(0)\cap B'} h(y)$$

for some constant $C > 0$. This implies $u$ is a harmonic function of order $O(r)$. The only harmonic functions that satisfy this are the affine functions, but the only affine function that makes $\tilde{h} = h + u$ bounded below is the constant function. Therefore, $u$ must be constant. □

*Remark* 3.5. Although in Lemma 3.4(a) we used the supremum norm, estimates for the derivatives can be obtained using elliptic regularity estimates. For example, when $B = \mathbb{R}^3$, the map $h(x)-\alpha-\frac{2|p_i|-4}{2r}$ is a harmonic function on the asymptotic region. According to the weighted Schauder estimate from Bartnik (1986) Proposition 1.6, for each $k \in \mathbb{N}$ there exists a $C > 0$ such that

$$\left\| r^{k+2}\, \nabla^k \left( h(x) - \alpha - \frac{2|p_i| - 4}{2r} \right) \right\|_{C^0} \leq C \left\| r^2 \left( h(x) - \alpha - \frac{2|p_i| - 4}{2r} \right) \right\|_{C^0} < \infty.$$

This implies that $\nabla^k \left( h(x) - \alpha - \frac{2|p_i|-4}{2r} \right) = \mathcal{O}(r^{-2-k})$ for all $k$.

### Circle bundle and involution

Using the existence of the harmonic function $h$, we can study circle bundles $P$ over $B'$. According to Chern (1977) all principal $S^1$-bundles are classified by the first Chern class. That is, for each element in $\sigma \in H^2(B', \mathbb{Z})$, there is a unique principal $S^1$-bundle $P \to B'$ and a (non-unique) connection[9] $\eta$ such that $\sigma = \frac{-1}{2\pi}[\mathrm{d}\,\eta]$. The Gibbons-Hawking ansatz requires $*\,\mathrm{d}\,h = \mathrm{d}\,\eta$ and hence it is sufficient to show $*\,\mathrm{d}\,h$ is represented by an element in the integral cohomology class. We already know $h$ is harmonic and hence $*\,\mathrm{d}\,h$ is closed and induces an element in $H^2_{dR}(B')$.

[9] The factor $-1/2\pi$ is due to the identification of $\mathfrak{u}(1) = i\mathbb{R}$ with $\mathbb{R}$.

**Lemma 3.6.** *For all $\Sigma \in H_2(B')$, we have that $\frac{-1}{2\pi}\int_\Sigma * \mathrm{d}\, h \in \mathbb{Z}$ and $* \mathrm{d}\, h$ is represented by an element in $H^2(B', \mathbb{Z})$.*

*Proof.* First we need to calculate the second cohomology class of $B'$. Notice that $B$ can be reconstructed from $B'$ by adding a certain number of balls that are centred around the singularities, and hence the topology of $B'$ can be determined using the Mayer-Vietoris sequence. The relevant part of this sequence is

$$0 \to H^2(B, \mathbb{Z}) \to H^2(B', \mathbb{Z}) \to \mathbb{Z}^{2\cdot\#\{p_i\}+\#\{q_j\}} \to 0.$$

Using the explicit topology of $B \in \{\mathbb{R}^3, \mathbb{R}^2 \times S^1, \mathbb{R} \times T^2\}$, we conclude

$$H_2(B') = \begin{cases} \mathbb{Z}^{2\cdot\#\{p_i\}+\#\{q_j\}} & \text{if } B = \mathbb{R}^3 \\ \mathbb{Z}^{2\cdot\#\{p_i\}+\#\{q_j\}} & \text{if } B = \mathbb{R}^2 \times S^1 \\ \mathbb{Z}^{2\cdot\#\{p_i\}+\#\{q_j\}+1} & \text{if } B = \mathbb{R} \times T^2. \end{cases}$$

Most elements of $H_2(B')$ are given by the 2-spheres centred around the singularities $\pm p_i$ and $q_j$. When $B = \mathbb{R}\times T^2$, there is one extra cycle which is the 2-torus at infinity. Using Lemma 3.4(b) we have explicit estimates for $* \mathrm{d}\, h$ near the singularities. Using this and the fact that $\int_{S^2(\pm p_i)} * \mathrm{d}\, h$ must be radially independent, we conclude that

$$\frac{-1}{2\pi}\int_{S^2(\pm p_i)} * \mathrm{d}\, h = 1, \qquad \frac{-1}{2\pi}\int_{S^2(q_j)} * \mathrm{d}\, h = -4.$$

Finally, we need to calculate $\int_{T^2} * \mathrm{d}\, h$ over the 2-torus for the case $B = \mathbb{R} \times T^2$. We use a similar idea as in Charbonneau & Hurtubise (2011) Proposition 3.5: Pick some $x > 0$ sufficiently large and consider the integral $\frac{-1}{2\pi}\int_{[-x,x]\times T^2} \mathrm{d} * \mathrm{d}\, h$. This integral must vanish due to the harmonicity of $h$. The boundary of $[-x, x] \times T^2 \subset B'$ decomposes into

$$\{\pm x\} \times T^2 \bigsqcup \sqcup_i S^2(\pm p_i) \bigsqcup \sqcup_j S^2(q_j),$$

and hence Stoke's theorem implies

$$0 = \int\limits_{\{x\}\times T^2} * \mathrm{d}\, h + \int\limits_{\{-x\}\times T^2} * \mathrm{d}\, h - \sum_i \int\limits_{S^2(\pm p_i)} * \mathrm{d}\, h - \sum_j \int\limits_{S^2_\delta(q_j)} * \mathrm{d}\, h.$$

When we impose the $\mathbb{Z}_2$ invariance of $h$,

$$2 \cdot \frac{-1}{2\pi} \int_{\{x\}\times T^2} * \mathrm{d}\, h = 4|q_i| - 2|p_i| = 16 - 2|p_i| \in 2\mathbb{Z}.$$

This concludes the first part of this lemma. In order to conclude that $[* \mathrm{d}\, h]$ is induced by an element in $H^2(B', \mathbb{Z})$, we notice that $H_2(B')$ has no torsion. Therefore the map $H^2(B, \mathbb{Z}) \to H^2_{dR}(B')$ is injective and its image contains all $[\sigma] \in H^2_{dR}(B')$ such that $\frac{-1}{2\pi} \int_\Sigma \sigma \in \mathbb{Z}$. □

**Definition 3.7.** *Let $h$ be the harmonic function defined in Lemma 3.4. The principal circle bundle $P$ that satisfies $c_1(P) = [* \mathrm{d}\, h]$ will be referred to as the principal bundle.*

Similarly to what Sen did, we want to lift the antipodal map $\tau$ to a free $\mathbb{Z}_2$ action $\tilde{\tau}$ on $P$. In order to glue in Atiyah-Hitchin at the fixed points $q_j$, we need to require that $\tilde{\tau}(e^{i\phi} \cdot p) = e^{-i\phi} \cdot \tilde{\tau}(p)$ for all $p \in P$ and $\phi \in \mathbb{R}$. We claim that such a lift exists.

**Lemma 3.8.** *There exists a lift $\tilde{\tau}$ of $\tau$ on $P$ such that for all $p \in P$ and $\phi \in \mathbb{R}$,*

$$\tilde{\tau}(e^{i\phi} \cdot p) = e^{-i\phi} \cdot \tilde{\tau}(p),$$

*and this lift is unique up to gauge transformation.*

*Proof.* Using the explicit bijection between principal $S^1$-bundles and $H^2(B', \mathbb{Z})$ in the proof of Chern (1977), one can show $\tilde{\tau}$ exists if and only if

$$\tilde{\tau} * c_1(P) = -c_1(P).$$

This is satisfied, because the harmonic function $h$ is invariant under $\tau$.

In order to show uniqueness, assume that there are two lifts $\tilde{\tau}_1$ and $\tilde{\tau}_2$. For any $p \in P$, the involutions $\tilde{\tau}_1$ and $\tilde{\tau}_2$ map $p$ to the same fiber and hence there is a $\phi_p \in \mathbb{R}$ such that $\tilde{\tau}_1(p) = e^{i\phi_p} \cdot \tilde{\tau}_2(p)$. Hence, these lifts only differ by gauge transformation. □

**Definition 3.9.** *Unless specified otherwise, we refer to $\tilde{\tau}$ from Lemma 3.8 as the $\mathbb{Z}_2$ action on $P$.*

Let $\eta$ be a connection on the principal bundle $P$. Because $c_1(P) = [* \mathrm{d}\, h]$, we can pick $\eta$ such that it satisfies the Bogomolny equation $* \mathrm{d}\, h = \mathrm{d}\, \eta$. With this information we are able to apply the Gibbons-Hawking Ansatz and $h\, g_B + h^{-1}\eta^2$ is a hyperkähler metric on $P$. We also have a free $\mathbb{Z}_2$ action on our principal bundle. If we make $\eta$ antisymmetric under $\tilde{\tau}$, then the Gibbons-Hawking metric will be invariant under this involution and the Gibbons-Hawking metric descends to an hyperkähler metric on $P/\mathbb{Z}_2$, which we pick as our bulk space.

## 3.2 The parameter space of $\eta$

Before we continue with the gluing construction, let us first determine the freedom we have in choosing $\eta$. Let $\eta$ and $\tilde{\eta}$ be two antisymmetric connections on $P$ satisfying the same Bogomolny equation $* \mathrm{d}\, h = \mathrm{d}\, \eta = \mathrm{d}\, \tilde{\eta}$. Their difference must be a pull-back of a closed 1-form on the base space. If this 1-form is exact, the connections differ by a gauge transformation, in which case the induced Gibbons-Hawking metrics $hg_B + h^{-1}\eta^2$ are isometric. Therefore, we only consider $\tilde{\eta} - \eta$ up to exact form and view it as an element of $H^1(B')$. By the Mayer-Vietoris sequence, $H^1(B') = H^1(B)$, and so

$$[\tilde{\eta} - \eta] = \begin{cases} 0 & \text{if } B = \mathbb{R}^3 \\ a\, [\mathrm{d}\, z] & \text{if } B = \mathbb{R}^2 \times S^1 \\ a\, [\mathrm{d}\, y] + b\, [\mathrm{d}\, z] & \text{if } B = \mathbb{R} \times T^2 \end{cases}$$

where $a, b \in \mathbb{R}$. If $a$ and $b$ are integers, then the gauge transformations $e^{i\, az}$ and $e^{i(ay+bz)}$ are well-defined and identify $\eta$ with $\tilde{\eta}$. If $a$ and $b$ are not integers, then $\eta$ and $\tilde{\eta}$ are not gauge-equivalent connections, as their holonomies at infinity differ. We conclude,

**Proposition 3.10.** *When $B = \mathbb{R}^3$, the connection $\eta$ is uniquely determined up to gauge transformation. When $B \neq \mathbb{R}^3$, the connection is uniquely determined up to an element in $H^1(B, \mathbb{R})/H^1(B, \mathbb{Z})$.*

An alternative way of studying the degrees of freedom in $\eta$ is to consider the connection it induces at infinity: According to Lemma 3.4, there is some constant $c \in \mathbb{R}$ such that

$$\mathrm{d}\,\eta = *\,\mathrm{d}\,h = \begin{cases} -\frac{c}{2} \cdot \mathrm{Vol}_{S^2} + \mathcal{O}(r^{-4}) & \text{if } B = \mathbb{R}^3 \\ c \cdot \mathrm{Vol}_{T^2} + \mathcal{O}(r^{-3}) & \text{if } B = \mathbb{R}^2 \times S^1 \\ c \cdot \mathrm{Vol}_{T^2} + \mathcal{O}(e^{-r}) & \text{if } B = \mathbb{R} \times T^2. \end{cases}$$

According to Lemma 3.6, the closed 2-forms $-\frac{c}{2}\cdot\mathrm{Vol}_{S^2}$ and $c\cdot\mathrm{Vol}_{T^2}$ are representatives of elements in $H^2(S^2, \mathbb{Z})$ and $H^2(T^2, \mathbb{Z})$ respectively and hence there is a connection $\eta_\infty$ on an circle bundle over $S^2$ or $T^2$ such that

$$\mathrm{d}\,\eta = \mathrm{d}\,\eta_\infty + \begin{cases} \mathcal{O}(r^{-4}) & \text{if } B = \mathbb{R}^3 \\ \mathcal{O}(r^{-3}) & \text{if } B = \mathbb{R}^2 \times S^1 \\ \mathcal{O}(e^{-r}) & \text{if } B = \mathbb{R} \times T^2. \end{cases}$$

Because $\mathrm{d}\,\eta$ and $\mathrm{d}\,\eta_\infty$ represent the same element in $H^2$, there is a 1-form $\tilde{\eta}_\infty$ on the asymptotic region of $B$, such that $\mathrm{d}\,\eta = \mathrm{d}\,\eta_\infty + \mathrm{d}\,\tilde{\eta}_\infty$. By the following version of the Poincaré lemma, we can pick $\tilde{\eta}_\infty$ with an explicit decay rate:

**Lemma 3.11.** *Let $\Sigma$ be a compact $n$-dimensional manifold and consider $U = \mathbb{R} \times \Sigma$. Let $\tau$ be a closed $k$-form such that at some $r_0 \in \mathbb{R}$, $\tau|_{\{r_0\}\times\Sigma} = 0$. Then the radial integrand*

$$\tilde{\eta} = \int_{s\in(r_0,r)} (\iota_{\partial_r}\tau)\,\mathrm{d}\,s$$

*satisfies $\mathrm{d}\,\tilde{\eta} = \tau$.*

*Proof.* Calculate $\mathrm{d}\,\eta$ in local coordinates and apply the fundamental theorem of calculus. □

Using remark 3.5 with the explicit integration in Lemma 3.11, we conclude that for all $k \in \mathbb{N}$,

$$\nabla^k \tilde{\eta}_\infty = \begin{cases} r^{-3-k} & \text{if } B = \mathbb{R}^3 \\ r^{-2-k} & \text{if } B = \mathbb{R}^2 \times S^1 \\ e^{-r} & \text{if } B = \mathbb{R} \times T^2 \end{cases}$$

with respect to the Euclidean metric on $B$.

By construction, the difference between $\eta$ and $\eta_\infty + \tilde{\eta}_\infty$ is closed. The space of flat $U(1)$-connections on a compact 2-manifold $\Sigma$ is parametrised by $H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$, and hence we can pick $\eta_\infty$ such that $\eta - \eta_\infty - \tilde{\eta}_\infty$ is exact. In summary,

**Lemma 3.12.** *Let $r$ be the Euclidean distance from the origin on $\mathbb{R}^3$, $\mathbb{R}^2$ or $\mathbb{R}$ when $B = \mathbb{R}^3$, $B = \mathbb{R}^2 \times S^1$ or $B = \mathbb{R} \times T^2$ respectively. Far away from the singularities, there exists an $r$-independent connection $\eta_\infty$ on a $S^1$-bundle over a compact set and a 1-form $\tilde{\eta}_\infty$ on the asymptotic region of the base space such that*

$$\eta = \eta_\infty + \tilde{\eta}_\infty$$

*up to gauge transformation. With respect to $g_B$,*

$$\nabla^k \tilde{\eta}_\infty = \begin{cases} \mathcal{O}(r^{-3-k}) & \text{if } B = \mathbb{R}^3 \\ \mathcal{O}(r^{-2-k}) & \text{if } B = \mathbb{R}^2 \times S^1 \\ \mathcal{O}(e^{-r}) & \text{if } B = \mathbb{R} \times T^2 \end{cases}$$

*for all $k \geq 0$.*

When $B = \mathbb{R}^3$, the connections $\eta$ and $\eta_\infty$ are uniquely determined up to gauge transformation. Similarly, when $B = \mathbb{R} \times T^2$, both $\eta$ and $\eta_\infty$ can both be chosen from a 2-parameter family. However, in the case $B = \mathbb{R}^2 \times S^1$ we have two degrees of freedom for the asymptotic connection, while according to Proposition 3.10 we only have one for $\eta$. In order to explain this apparent discrepancy, we describe the circle bundle over the asymptotic region of $\mathbb{R}^2 \times S^1$ with explicit coordinates. In these coordinates we give the asymptotic connection explicitly and solve the apparent discrepancy by considering the holonomy.

Consider $\mathbb{R}^3$ with the coordinates $\{\phi, z, t\}$ and quotient it with the free $\mathbb{Z}^3$ action

$$(n, m, p) \cdot (\phi, z, t) = (\phi + 2\pi\, n, z + 2\pi\, m, t + 2\pi\, p + 2\pi c\, m\, \phi), \tag{8}$$

where $c$ is a fixed integer. Under the projection $(\phi, z, t) \mapsto [\phi, z] \in T^2$, the space $\mathbb{R}^3/\mathbb{Z}^3$ becomes an $S^1$-bundle over $T^2$. For any $a, b \in \mathbb{R}$ the 1-form

$$\eta_\infty = \mathrm{d}\, t - c\, z\, \mathrm{d}\, \phi + a\, \mathrm{d}\, \phi + b\, \mathrm{d}\, z$$

is invariant under the $\mathbb{Z}^3$ action and descends to a connection on $\mathbb{R}^3/\mathbb{Z}^3$ satisfying

$\mathrm{d}\,\eta_\infty = c\,\mathrm{Vol}_{T^2}$. The values $a, b \in \mathbb{R}/\mathbb{Z}$ are the variables by which the space of connections on $T^2$ is parametrised. One can calculate the holonomy along $S^1_\phi \times \{z_0\}$ and $\{\phi_0\} \times S^1_z \subseteq T^2$, which is $e^{2\pi i\cdot(c\,z_0 - a)}$ and $e^{-2\pi i\cdot(c\,\phi_0+b)}$ respectively.

We focus now on the case when at infinity the connection $\eta_\infty$ is identified with the connection $\eta$ that was constructed by the Gibbons-Hawking ansatz. By calculating the holonomy of $\eta$, we can retrieve the values of $a$ and $b$. As explained in Proposition 3.10, any value for $b \in \mathbb{R}/\mathbb{Z}$ can be attained. We claim however, that $a$ must be zero. Indeed, consider the loop $\gamma$ along the circle in the punctured plane $B'|_{z=0}$ and consider the holonomy along this circle when the radius $r_0$ is very large. The principal bundle over the punctured plane $B'|_{z=0}$ is trivial and hence there is an $\eta_0 \in \Omega^1(B'|_{z=0})$ such that $\eta$ trivialises as $\mathrm{d}\,t + \eta_0$. The holonomy along $\gamma$ is given by

$$\mathrm{Hol}(\gamma) = -\int_\gamma \eta_0 \in \mathbb{R}/2\pi\mathbb{Z}.$$

Let $Ann$ be the region in $B'|_{z=0}$ that is bounded by $\gamma$. That is, $Ann$ is a punctured plane in $B$ with boundaries $\gamma$, $\gamma_{q_0}$ and $\gamma_{\pm p_{i'}}$ where $\gamma_{q_0}$ and $\gamma_{\pm p_{i'}}$ are arbitrary small loops around the singularities that lie in the plane $z = 0$. By Stoke's theorem and the Bogomolny equation,

$$\mathrm{Hol}(\gamma) = -\int_\gamma \eta_0 = -\int_{Ann} *\,\mathrm{d}\,h - \int_{\gamma_{q_0}} \eta_0 - \sum_{i'} \int_{\gamma_{\pm p_{i'}}} \eta_0.$$

Because the harmonic function $h$ is invariant under the antipodal map, $*\,\mathrm{d}\,h$ is anti-symmetric and hence $\int_A *\,\mathrm{d}\,h$ vanishes. To consider the contribution $\int_{\gamma_{q_0}} \eta_0$, we consider the upper half sphere $S^+(q_0)$ with boundary $\gamma_{q_0}$. According to lemma 3.4, $h \simeq a_j - \frac{2}{|x-q_0|} + \mathcal{O}(|x-q_0|^2)$, which implies

$$\int_{\gamma_{q_0}} \eta_0 = \int_{S^+(q_0)} *\,\mathrm{d}\,h = \int_{S^+(q_0)} 2\,\mathrm{Vol}(S^2) + \mathcal{O}(r) = 4\pi.$$

A similar calculation can be done for $\int_{\gamma_{\pm p_{i'}}} \eta_0$. Because the non-fixed points come in pairs, their contribution will again be a multiple of $2\pi$, and hence

$$-2\pi \cdot a \in 2\pi\mathbb{Z}.$$

Because $\mathrm{Hol}(\gamma) \in \mathbb{R}/2\pi\mathbb{Z}$, we conclude $a = 0$. A similar study is done for the case $B = \mathbb{R} \times T^2$ by Hein et al. (2022), and hence in general we have:

**Proposition 3.13.** *When $B = \mathbb{R}^2 \times S^1$ or $B = \mathbb{R} \times T^2$, there exists a unique connection $\eta_\infty$ on an $S^1$-bundle over $T^2$ such that asymptotically*

$$\eta = \eta_\infty + \begin{cases} \mathcal{O}(r^{-2}) & \text{if } B = \mathbb{R}^2 \times S^1 \\ \mathcal{O}(e^{-r}) & \text{if } B = \mathbb{R} \times T^2 \end{cases}$$

*up to a gauge transformation. Moreover, in the coordinates $\{r, \phi, z\}$ or $\{r, y, z\}$ on $\mathbb{R}^2 \times S^1$ or $\mathbb{R} \times T^2$ respectively, the bundle $P|_{r=\infty}$ can be trivialized such that*

$$\eta_\infty = \begin{cases} \mathrm{d}\, t - c\, z\, \mathrm{d}\, \phi + b\, \mathrm{d}\, z & \text{if } B = \mathbb{R}^2 \times S^1 \\ \mathrm{d}\, t - c\, z\, \mathrm{d}\, y + a\, \mathrm{d}\, y + b\, \mathrm{d}\, z & \text{if } B = \mathbb{R} \times T^2 \end{cases}$$

*for some $a, b \in [0, 2\pi)$ and $c \in \mathbb{Z}$.*

When $P|_{r=\infty}$ is non-trivial, one can use rotations and translations to set the constants $a$ and $b$ in Proposition 3.13 to arbitrary values. For example, in the case $B = \mathbb{R}^2 \times S^1$ with coordinates $\{r, \phi, z\}$, fix $\phi_0, z_0 \in \mathbb{R}$ and consider the path $\gamma(s) = (r, \phi + \phi_0\, s, z + z_0\, s)$ on $B$. In the coordinates used in Equation 8, the horizontal lift of this path at infinity can be chosen such that

$$\gamma(1) = (\phi + \phi_0, z + z_0, t + c\, \phi_0 \cdot z).$$

This horizontal lift induces a diffeomorphism on $P$ which is isometric on $B$. Moreover,

$$\gamma^* \eta_\infty - \eta_\infty = c\phi_0 \,\mathrm{d}\, z - c z_0 \,\mathrm{d}\, \phi.$$

Hein et al. (2022) also made this observation and argued that ALH manifolds will have $a$ and $b$ as parameters for their moduli space, but ALH* does not. We expect a similar behaviour in our ALG*/ALG case. Namely, the action of $\gamma$ can be viewed as a rotation on the $\mathbb{R}^2$ plane in $B = \mathbb{R}^2 \times S^1$. Under these rotations one can map $\eta_\infty$ to the fixed connection $\mathrm{d}\, t - cz \,\mathrm{d}\, \phi$. Moreover, this transformation acts on the Kähler forms by a hyperkähler rotation. By rotating the base of $\mathbb{R}^3/L$ in the opposite direction simultaneously, one can keep the Kähler forms fixed while modifying $\eta_\infty$.

## 3.3 The collapsing parameter

We return to the construction of the bulk space. The second step of the gluing procedure is to pick a collapsing parameter. From the Gibbons-Hawking construction there are two obvious parameters to choose:

- The constant in the harmonic function $h$.
- The global scale of the metric.

Indeed, according to Lemma 3.4, the function $h$ is uniquely determined up to a constant, and for any $C > 0$, the function $C + h$ satisfies the Bogomolny equation for the same connection. Hence $C$ is a free parameter which can be varied in our construction. Similarly, given an hyperkähler metric $g$, the metric $C \cdot g$ is also hyperkähler. Although these parameters look independent, they are actually related by a coordinate transformation. Indeed, trivialise the metric on the base space $g_B = \mathrm{d}\,x_1^2 + \mathrm{d}\,x_2^2 + \mathrm{d}\,x_3^2$ and consider the Gibbons-Hawking metric

$$g = (C + h)\,\mathrm{d}\,x_i^2 + \frac{1}{C+h}\eta^2.$$

Next consider the transformation $\tilde{x}_i = Cx_i$ and denote $\tilde{g}_B = \mathrm{d}\,\tilde{x}_1^2 + \mathrm{d}\,\tilde{x}_2^2 + \mathrm{d}\,\tilde{x}_3^2$. In these new coordinates the metric reads

$$g = C^{-2}(C + h)\tilde{g}_B + \frac{1}{C+h}\eta^2.$$

Comparing the hodge dual for $g_B$ and $\tilde{g}_B$, one can show $*^{\tilde{g}_B}\,\mathrm{d}(1+C^{-1}h) = \mathrm{d}\,\eta$. This implies that the metric

$$C \cdot g = (1 + C^{-1}h)\tilde{g}_B + \frac{1}{1+C^{-1}h}\eta^2$$

also arises from the Gibbons-Hawking ansatz. Moreover, the Green's function on $B$ transforms as $G_{g_B}(x, x_0) = C \cdot G_{\tilde{g}_B}(\tilde{x}, C \cdot x_0)$ and so the rescaling of the metric can be viewed as a combination of a rescaling of the lattice $L$, a translation of the singularities and a change of constant.

Because changing the harmonic function and rescaling induce equivalent metrics, we pick our collapsing parameter as a combination of them. We choose our metric such that for any point on $B'$, the length of the fiber converges to $2\pi\epsilon$ as our collapsing parameter $\epsilon$ tends to zero:

**Definition 3.14.** *Consider $P$, $\tilde{\tau}$ from definitions 3.7 and 3.9 and $h$ from Lemma 3.4. Let $\eta$ be a antisymmetric connection on $P$ such that $*\mathrm{d}\, h = \mathrm{d}\,\eta$. For any $\epsilon > 0$ we define the harmonic function*

$$h_\epsilon = 1 + \epsilon h.$$

*From now on, the metric that is induced by the Gibbons-Hawking ansatz with the harmonic function $\epsilon^{-1} + h$ and $\eta$, and is rescaled by a factor of $\epsilon$ will be called the Gibbons-Hawking metric and denoted as $g^{GH}$. Explicitly, it is given by*

$$g^{GH} = h_\epsilon g_B + \frac{\epsilon^2}{h_\epsilon}\eta^2$$

*and its Kähler forms are*

$$\begin{aligned}\omega_1^{GH} =&\epsilon\,\mathrm{d}\,x_1 \wedge \eta + h_\epsilon\,\mathrm{d}\,x_2 \wedge \mathrm{d}\,x_3\\ \omega_2^{GH} =&\epsilon\,\mathrm{d}\,x_2 \wedge \eta + h_\epsilon\,\mathrm{d}\,x_3 \wedge \mathrm{d}\,x_1\\ \omega_3^{GH} =&\epsilon\,\mathrm{d}\,x_3 \wedge \eta + h_\epsilon\,\mathrm{d}\,x_1 \wedge \mathrm{d}\,x_2.\end{aligned}$$

*The hyperkähler space $(P/\mathbb{Z}_2, g^{GH}, \omega_i^{GH})$ will be called the bulk space.*

## 3.4 Gauge fixing and the gluing of Kähler forms

In the third step of the gluing construction we make the bulk space complete and equip it with an almost hyperkähler metric. To do this, we have to identify the asymptotic regions of the Atiyah-Hitchin manifolds and the Taub-NUT spaces to a neighbourhood of the singularities. Due to our choice of harmonic function $h$, this can be done explicitly. For example, on the tubular neighbourhood of $P$ near a fixed point singularity $q_j$, the Gibbons-Hawking ansatz yields a circle bundle over $\mathbb{R} \times S^2$ of degree $-4$. As explained before, the asymptotic region of the branched double cover of the Atiyah-Hitchin manifold has the same topology. Moreover, on this neighbourhood, both the Atiyah-Hitchin metric and the metric on the bulk space approximate the Taub-NUT metric with mass $-4$ and their involutions co-inside.

Instead of interpolating the metrics we will interpolate the Kähler forms. In order to get the correct error estimates, we have to modify the diffeomorphism between the bulk space $P/\mathbb{Z}_2$ and the asymptotic regions of the Atiyah-Hitchin manifolds and Taub-NUT spaces using a suitable gauge transformation. We explain our choice of gauge transformation and we give the interpolated forms explicitly.

Because the Taub-NUT metric with negative mass is a suitable model for the Atiyah-Hitchin manifold, we will use it to measure the errors on the gluing region. Similarly, we will use the standard Taub-NUT metric as the model metric for the non-fixed singularities. These metrics will be used later in the thesis and hence we will fix them once and for all:

**Definition 3.15.** *Let $p_i$ be a non-fixed singularity and let $r_i$ be the distance to $p_i$ on $B$. Let $\alpha_i > 0$ be such that, near $p_i$, $h(x) = \alpha_i + \frac{1}{2|x-p_i|} + \mathcal{O}(|x-p_i|)$. For the model metric near $p_i$ define*

$$h^{p_i} := \alpha_i + \frac{2}{r_i}, \qquad \rho_{p_i} := \log r_i,$$
$$h^{p_i}_\epsilon := 1 + \epsilon\, h^{p_i}, \qquad \Omega_{p_i} := r_i^{-1} \left(h^{p_i}_\epsilon\right)^{-\frac{1}{2}}.$$

*Let $U^{p_i} \subseteq B'$ be a punctured neighbourhood of $p_i$ homotopic to $S^2$ and let $\eta^{p_i}$ be an $r_i$-invariant connection of $P|_{U^{p_i}}$ satisfying the Bogomolny equation*

$$* \operatorname{d} h^{p_i} = \operatorname{d} \eta^{p_i}.$$

*Define $g^{p_i}$ to be the Gibbons-Hawking metric induced by $h^{p_i}$ and $\eta^{p_i}$, i.e.*

$$g^{p_i} := h^{p_i}_\epsilon g_{U^{p_i}} + \frac{\epsilon^2}{h^{p_i}_\epsilon} (\eta^{p_i})^2.$$

*We call the hyperkähler manifold $(P|_{U^{p_i}}, g^{p_i})$ the model space near $p_i$. Its Kähler forms are*

$$\omega_1^{p_i} = \epsilon \operatorname{d} x_1 \wedge \eta^{p_i} + h^{p_i}_\epsilon \operatorname{d} x_2 \wedge \operatorname{d} x_3$$
$$\omega_2^{p_i} = \epsilon \operatorname{d} x_2 \wedge \eta^{p_i} + h^{p_i}_\epsilon \operatorname{d} x_3 \wedge \operatorname{d} x_1$$
$$\omega_3^{p_i} = \epsilon \operatorname{d} x_3 \wedge \eta^{p_i} + h^{p_i}_\epsilon \operatorname{d} x_1 \wedge \operatorname{d} x_2.$$

*Also define the conformally rescaled model metric*

$$g^{p_i}_{cf} := \Omega^2_{p_i} g^{p_i} = \operatorname{d} \rho^2_{p_i} + g_{S^2} + \frac{\epsilon^2}{(h^{p_i}_\epsilon)^2} (\eta^{p_i})^2.$$

**Definition 3.16.** *Let $q_j$ be a fixed point singularity and let $r_j$ be the distance to $q_j$ on $B$. Let $\alpha_j > 0$ be such that, near $q_j$, $h(x) = \alpha_j - \frac{2}{|x-q_j|} + \mathcal{O}(|x-q_j|^2)$. For the model metric near $q_j$ define*

$$\begin{aligned} h^{q_j} :=& \alpha_j - \frac{2}{r_j}, & \rho_{q_j} :=& \log r_j, \\ h_\epsilon^{q_j} :=& 1 + \epsilon\, h^{q_j}, & \Omega_{q_j} :=& r_j^{-1}\,(h_\epsilon^{q_j})^{-\frac{1}{2}}. \end{aligned}$$

*Let $U^{q_j} \subseteq B'$ be a punctured neighbourhood of $q_j$ homotopic to $S^2$ and let $\eta^{q_j}$ be an $r_j$-invariant connection of $P|_{U^{q_j}}$ satisfying the Bogomolny equation*

$$* \,\mathrm{d}\, h^{q_j} = \mathrm{d}\,\eta^{q_j}.$$

*Define $g^{q_j}$ to be the Gibbons-Hawking metric induced by $h^{q_j}$ and $\eta^{q_j}$, i.e.*

$$g^{q_j} := h_\epsilon^{q_j} g_{U^{q_j}} + \frac{\epsilon^2}{h_\epsilon^{q_j}}(\eta^{q_j})^2.$$

*We call the hyperkähler manifold $(P|_{U^{q_j}}, g^{q_j})$ the model space near $q_j$. Its Kähler forms are*

$$\begin{aligned} \omega_1^{q_j} =& \epsilon\,\mathrm{d}\,x_1 \wedge \eta^{q_j} + h_\epsilon^{q_j}\,\mathrm{d}\,x_2 \wedge \mathrm{d}\,x_3 \\ \omega_2^{q_j} =& \epsilon\,\mathrm{d}\,x_2 \wedge \eta^{q_j} + h_\epsilon^{q_j}\,\mathrm{d}\,x_3 \wedge \mathrm{d}\,x_1 \\ \omega_3^{q_j} =& \epsilon\,\mathrm{d}\,x_3 \wedge \eta^{q_j} + h_\epsilon^{q_j}\,\mathrm{d}\,x_1 \wedge \mathrm{d}\,x_2. \end{aligned}$$

*Also define the conformally rescaled model metric*

$$g_{cf}^{q_j} := \Omega_{q_j}^2 g^{q_j} = \mathrm{d}\,\rho_{q_j}^2 + g_{S^2} + \frac{\epsilon^2}{(h_\epsilon^{q_j})^2}(\eta^{q_j})^2.$$

*Remark* 3.17. The choice and use of the conformal rescaling will be explained in Chapter 4. Namely, it turns out that this conformal rescaling will be a useful tool in understanding the analytical properties of the Laplacian. Moreover, one can view the $C^k$ norms with respect to $g_{cf}^{p_i}$ and $g_{cf}^{q_j}$ as a generalisation of the norms in Bartnik (1986). For example, when $u\colon U^{q_j} \to \mathbb{R}$ is a function for which $r^l \nabla_{g_B}^l u$ is bounded for all $l \leq k$, then its induced function on the principal bundle $P|_{U^{q_j}}$ has a bounded

$C^k$ norm with respect to $g_{cf}^{q_j}$. Indeed, the covariant derivatives w.r.t. $g_{cf}^{q_j}$ are given by

$$\begin{aligned}
\mathrm{d}\, u =& \frac{\partial u}{\partial x_i}\,\mathrm{d}\, x_i \\
\nabla_{g_{cf}^{q_j}}\,\mathrm{d}\, u =& \frac{\partial^2 u}{\partial x_i \partial x_j}\,\mathrm{d}\, x_i \otimes \mathrm{d}\, x_j + \frac{\partial u}{\partial x_i}\nabla_{g_{cf}^{q_j}}\,\mathrm{d}\, x_i \\
\nabla^2_{g_{cf}^{q_j}}\,\mathrm{d}\, u =& \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}\,\mathrm{d}\, x_i \otimes \mathrm{d}\, x_j \otimes \mathrm{d}\, x_k + 2\frac{\partial^2 u}{\partial x_i \partial x_j}\nabla_{g_{cf}^{q_j}}\,\mathrm{d}\, x_i \otimes \partial x_j + \frac{\partial u}{\partial x_i}\nabla^2_{g_{cf}^{q_j}}\,\mathrm{d}\, x_i \\
& \ldots \text{etc} \ldots
\end{aligned}$$

By assumption, $r\frac{\partial u}{\partial x_i}$, $r^2\frac{\partial^2 u}{\partial x_i \partial x_j}$, $\ldots$, $r^k\frac{\partial^k u}{\partial x_{i_1}\ldots\partial x_{i_k}}$ are bounded and hence $\|u\|_{C^k_{g_{cf}^{q_j}}}$ is finite if $r^{-1}\nabla^l_{g_{cf}^{q_j}}\,\mathrm{d}\, x_i$ is bounded for all $l \leq k$. By the Koszul formula one can estimate these terms explicitly[10], which proves our claim.

A second reason to use $g_{cf}^{p_i}$ and $g_{cf}^{q_j}$ is that higher derivatives will have the same growth/decay rate as the functions itself. For example, according to Remark 3.5, $\nabla^k(h - h^{q_j}) = \mathcal{O}(r_j^{2-k})$ with respect to the Euclidean metric on the base space. Converting this to the conformal metric, one concludes

$$\nabla^k_{cf}(h - h^{q_j}) \sim r^k \nabla^k_{g_B}(h - h^{q_j}) = \mathcal{O}(r_j^2).$$

Therefore, higher order estimates follow automatically from the $C^0$ estimate and we can omit them in our calculations.

Let us come back to the study of the Kähler forms. According to Definition 3.14, the difference between the Kähler forms of $g^{GH}$ and the model metrics are

$$\begin{aligned}
\omega_i^{GH} - \omega^{p_i} =& \epsilon\,\mathrm{d}\, x_i \wedge (\eta - \eta^{p_i}) + (h_\epsilon - h_\epsilon^{p_i})\,\mathrm{d}\, x_j \wedge \mathrm{d}\, x_k, \\
\omega_i^{GH} - \omega^{q_j} =& \epsilon\,\mathrm{d}\, x_i \wedge (\eta - \eta^{q_j}) + (h_\epsilon - h_\epsilon^{q_j})\,\mathrm{d}\, x_j \wedge \mathrm{d}\, x_k.
\end{aligned}$$

Therefore, we need to estimate the difference between the connection $\eta$ and the connection on the model spaces. For this we use the same argument as in Lemma 3.12. For example, to estimate the closed 2-form $\mathrm{d}(\eta - \eta^{q_j}) = *^B\,\mathrm{d}(h - h^{q_j})$, we use Remark 3.5 to conclude $\mathrm{d}(\eta - \eta^{q_j})$ and its derivatives are of order $r_j^3$ w.r.t. $g_{cf}^{q_j}$. Therefore, we can integrate $\mathrm{d}(\eta - \eta^{q_j})$ from $r_0 = 0$ using Lemma 3.11. This yields

[10]See the calculations in Lemma 4.8.

a 1-form $\tilde{\eta}^{q_j}$ such that

$$\mathrm{d}\,\eta = \mathrm{d}\,\eta^{q_j} + \mathrm{d}\,\tilde{\eta}^{q_j}.$$

Because $H^1(S^2) = 0$, the form $\eta - \eta^{q_j} - \tilde{\eta}^{q_j}$ is exact and hence we have:

**Lemma 3.18.** *On a small annulus around each fixed point singularity $q_j$, there exists a gauge transformation which identifies $\eta$ with $\eta^{q_j} + \tilde{\eta}^{q_j}$, where $\tilde{\eta}^{q_j}$ and all its derivatives are of order $r_j^3$ with respect to $g_{cf}^{q_j}$.*

*Similarly, on a small annulus around each non-fixed singularity $p_i$, there exists a gauge transformation which identifies $\eta$ with $\eta^{p_i} + \tilde{\eta}^{p_i}$, where $\tilde{\eta}^{p_i}$ and all its derivatives are of order $r_i^2$ with respect to $g_{cf}^{p_i}$.*

With this we estimate the difference between the Kähler forms of $g^{GH}$ and $g^{q_j}$. Using the estimates from Lemmas 3.4 and Lemma 3.18 their difference is given by $\epsilon\,\mathrm{d}\,x_i \wedge \mathcal{O}(r_j^3) + \mathcal{O}(\epsilon r_j^2)\,\mathrm{d}\,x_j \wedge \mathrm{d}\,x_k$. Because $r^{-1}\,\mathrm{d}\,x_i$ and its derivatives are bounded in $g_{cf}^{q_j}$,

$$\|\nabla^k(\omega^{GH} - \omega^{q_j})\|_{g_{cf}^{q_j}} = \mathcal{O}(\epsilon r_j^4)$$

for all $k \geq 0$. Similarly, one can estimate $\|\nabla^k(\omega^{GH} - \omega^{p_i})\|_{g_{cf}^{p_i}} = \mathcal{O}(\epsilon r_i^3)$ near a non-fixed singularity $p_i$. This enables us to apply the radial integration from Lemma 3.11 again to find:

**Lemma 3.19.** *On a small annulus around the non-fixed singularity $p_i$, there exists a smooth triple of 1-forms, which we denote by $\sigma^{p_i,GH}$, such that*

$$\omega^{GH} = \omega^{p_i} + \mathrm{d}\,\sigma^{p_i,GH}.$$

*The 1-forms $\sigma^{p_i,GH}$ and all its derivatives are of order $\mathcal{O}(\epsilon r_j^3)$ with respect to $g_{cf}^{p_i}$.*

*On a small annulus around the fixed point singularity $q_j$, there exists a smooth triple of 1-forms, which we denote by $\sigma^{q_j,GH}$, such that*

$$\omega^{GH} = \omega^{q_j} + \mathrm{d}\,\sigma^{q_j,GH}.$$

*The 1-forms $\sigma^{q_j,GH}$ and all its derivatives are of order $\mathcal{O}(\epsilon r_j^4)$ with respect to $g_{cf}^{q_j}$.*

Next we compare the Atiyah-Hitchin metric to the Taub-NUT metric with negative mass $-4$ explicitly. According to M. Atiyah & Hitchin (1988), the Atiyah-Hitchin metric has a radial parameter $r_{AH}$ and for large values of $r_{AH}$ the metric on the branched double cover is

$$g^{AH} = \left(1 - \frac{2}{r_{AH}}\right)(\mathrm{d}\, r_{AH}^2 + r_{AH}^2 g_{S^2}) + \left(1 - \frac{2}{r_{AH}}\right)^{-1} (\eta^{q_j})^2 + \mathcal{O}(e^{-r_{AH}}).$$

Similar estimates are true for the Kähler forms. By identifying $r_j := \frac{\epsilon}{1+\epsilon\alpha_j} r_{AH}$, where $\alpha_j$ is defined in Lemma 3.4, one can show

$$\begin{aligned}
\frac{\epsilon^2}{1+\epsilon\alpha_j} g^{AH}(r_{AH}) =& g^{q_j}(r_j) + \mathcal{O}(e^{-\frac{1+\epsilon\alpha_j}{\epsilon} r_j}),\\
\frac{\epsilon^2}{1+\epsilon\alpha_j} \omega^{AH}(r_{AH}) =& \omega^{q_j}(r_j) + \mathcal{O}(e^{-\frac{1+\epsilon\alpha_j}{\epsilon} r_j}).
\end{aligned}$$

Here the error is estimated using $g^{q_j}$. Converting this into $g_{cf}^{q_j}$ and applying the radial integration from Lemma 3.11 again, we conclude

**Lemma 3.20.** *On the asymptotic region of the Atiyah-Hitchin manifold, there exists a triple of 1-forms $\sigma^{q_j,AH}$ such that*

$$\frac{\epsilon^2}{1+\epsilon\alpha_j} \omega^{AH} = \omega^{q_j} + \mathrm{d}\, \sigma^{q_j,AH}$$

*and $\sigma^{q_j,AH}$ and all its derivatives are $\mathcal{O}(r_j^2 e^{-\frac{1+\epsilon\alpha}{\epsilon} r_j})$ with respect to $g_{cf}^{q_j}$.*

In a similar manner we can compare the Kähler forms for the model metric near $p_i$ with a rescaled version of a fixed Taub-NUT space. In this case, there is no exponentially decaying error term and we get the result:

**Lemma 3.21.** *By identifying $r_i := \frac{\epsilon}{1+\epsilon\alpha_i} r_{TN}$, where $\alpha_i$ is defined in Lemma 3.4,*

$$g^{p_i} = \frac{\epsilon^2}{1+\epsilon\alpha_i} g^{TN}, \text{ and } \omega^{p_i} = \frac{\epsilon^2}{1+\epsilon\alpha_i} \omega^{TN},$$

*where $g^{TN}$ is the fixed Taub-NUT space*

$$g^{TN} := \left(1 + \frac{1}{2r_{TN}}\right)(\mathrm{d}\, r_{TN}^2 + r_{TN}^2 g_{S^2}) + \frac{1}{1 + \frac{1}{2r_{TN}}} (\eta^{p_i})^2.$$

With these ingredients we can finally construct a complete manifold and equip it with a definite triple that is almost hyperkähler. Near the fixed point singularities we complete the bulk space using Atiyah-Hitchin manifolds. Near the non-fixed points, we glue in Taub-NUT spaces[11]. With this setup, we define the complete 4-dimensional manifold underlying our gravitational instantons.

**Definition 3.22.** *Let $n$ be the number of non-fixed points $p_i$ and $m$ be the number of fixed point singularities $q_j$, defined in Definition 3.2 resp. 3.1. Let $P/\mathbb{Z}_2$ be the bulk space. Identify the asymptotic region of the Atiyah-Hitchin manifold and the neighbourhoods of $q_j$ on $P/\mathbb{Z}_2$ with the $\mathbb{Z}_2$ quotient of the model space defined in Definition 3.16. Similarly, identify the asymptotic region of the Taub-NUT space and the neighbourhoods of $p_i$ on $P/\mathbb{Z}_2$ with the the model space defined in Definition 3.15. Consider the connected sum of $P/\mathbb{Z}_2$ with $m$ copies of the Atiyah-Hitchin manifold and $n$ copies of the Taub-NUT space. We call this space the global space and we denote it as $M_{B,n}$.*

In order to equip $M_{B,n}$ with a definite triple, let $\epsilon \in (0,1)$, $R_0, R_1 \in (0,\infty)$ be small. Assume that the gluing in the connected sum construction happens on the region $\bigcup_i B_{R_1}(p_i)\backslash B_{R_0}(p_i)$ and $\bigcup_j B_{R_1}(q_j)\backslash B_{R_0}(q_j)$. For each point $p_i$ and $q_j$, pick $\chi_\epsilon(x)$ be a family of smooth step functions on $B$ such that $\chi_\epsilon(x) = 0$ when $\|x-p_i\|_{g^B}, \|x-q_j\|_{g^B} \leq R_0$ and $\chi_\epsilon(x) = 1$ when $\|x-p_i\|_{g^B}, \|x-q_j\|_{g^B} \geq R_1$. We pick the following triple on the connected sum:

$$\omega = \begin{cases} \frac{\epsilon^2}{1+\epsilon\alpha_i}\omega^{TN} & \textit{if} \qquad \|x-p_i\|_{g^B} \leq R_0 \\ \frac{\epsilon^2}{1+\epsilon\alpha_j}\omega^{AH} & \textit{if} \qquad \|x-q_j\|_{g^B} \leq R_0 \\ \omega^{p_i} + \mathrm{d}\left(\chi_\epsilon \sigma^{p_i,GH}\right) & \textit{if} \quad R_0 \leq \|x-p_i\|_{g^B} \leq R_1 \\ \omega^{q_j} + \mathrm{d}\left[(1-\chi_\epsilon)\sigma^{q_j,AH} + \chi_\epsilon\sigma^{q_j,GH}\right] & \textit{if} \quad R_0 \leq \|x-q_j\|_{g^B} \leq R_1 \\ \omega_i^{GH} & \text{otherwise.} \end{cases}$$

We need to find $\chi_\epsilon$, $R_0$ and $R_1$ such that $\omega_i$ is hyperkähler outside $r \in [R_0, R_1]$ and behaves well enough inside. We will explain where our choices come from. Assume that $R_0 = C_0\epsilon^\kappa$ and $R_1 = C_1\epsilon^\kappa$ for some $C_0, C_1 > 0$, $\kappa \in \mathbb{R}$. We need to balance the following factors:

[11] Alternatively, one can complete the region near the non-fixed points in a similar way as the Taub-NUT space is completed by adding some extra points. To unify the gluing procedure for the Atiyah-Hitchin manifold and the Taub-NUT space we prefer to use the first method.

- For the approximations of $\sigma^{p_i,GH}$ and $\sigma^{q_j,GH}$ we need the radial distance to the singularity to be small. This is satisfied when $\kappa > 0$.
- At the same time we need that $h_\epsilon > 0$ and so $r_j$ cannot be too small. This is satisfied when $C_0 = 4$ and $\kappa < 1$, because Lemma 3.4 implies $h_\epsilon > 0$ if $4\epsilon^\kappa > 4\epsilon$.
- For the approximation of $\sigma^{q_j,AH}$ we need $r_{AH}$ to be large. Combining $r_j = \mathcal{O}(\epsilon^\kappa)$ and $r_{AH} = \frac{1+\epsilon\alpha_j}{\epsilon} r_j$, it follows $r_{AH} = \mathcal{O}(\epsilon^{\kappa-1})$. This is large when $\kappa < 1$.
- Finally we need that $R_0 < R_1$. This happens when $C_0 < C_1$.

From Lemma 3.19 and 3.20, we have decay estimates for $\sigma^{p_i,GH}$, $\sigma^{q_j,GH}$ and $\sigma^{q_j,AH}$. It is sufficient if we assume $\sigma^{p_i,GH}, \sigma^{q_j,GH} = \mathcal{O}(\epsilon r_j^3)$ and $\sigma^{q_j,AH} = \mathcal{O}(\epsilon^3 r_j^{-1})$. When we pick

$$R_0 = 4\epsilon^{\frac{2}{5}} \text{ and } R_1 = 5\epsilon^{\frac{2}{5}}$$

all the above requirements are satisfied. By estimating $\chi_\epsilon$, one notices that $\mathrm{d}\,\chi_\epsilon = \mathcal{O}(1)$ and may conclude.

**Theorem 3.23.** *There exists an $\epsilon_1 > 0$ such that for all $0 < \epsilon < \epsilon_1$:*

1. *$\omega_i$ is a closed 2-form in $M_{B,n}$.*
2. *Outside the gluing region (i.e. $r_i \in [4\epsilon^{\frac{2}{5}}, 5\epsilon^{\frac{2}{5}}]$ or $r_j \in [4\epsilon^{\frac{2}{5}}, 5\epsilon^{\frac{2}{5}}]$), $\omega_i$ is an hyperkähler triple.*
3. *Inside the gluing region near the fixed point $q_j$, $\omega_i - \omega_i^{q_j}$ and all its derivatives are of order $\mathcal{O}(\epsilon^3 r_j^{-1}) + \mathcal{O}(\epsilon r_j^3)$ w.r.t. $g_{cf}^{q_j}$. In particular, inside this gluing region $\omega_i$ is a definite triple of closed 2-forms such that*
$$\frac{1}{2}\omega_i \wedge \omega_j = \left(\mathrm{Id} + \mathcal{O}(\epsilon^{7/5})\right)_{ij} \otimes \mathrm{Vol}^{g^{q_j}} .$$
4. *Inside the gluing region near the non-fixed point $p_i$, $\omega_i - \omega_i^{p_i}$ and all its derivatives are of order $\mathcal{O}(\epsilon r_j^3)$ w.r.t. $g_{cf}^{p_i}$. In particular, inside this gluing region $\omega_i$ is a definite triple of closed 2-forms such that*
$$\frac{1}{2}\omega_i \wedge \omega_j = \left(\mathrm{Id} + \mathcal{O}(\epsilon^{7/5})\right)_{ij} \otimes \mathrm{Vol}^{g^{p_i}} .$$

## 3.5 Global properties of $M_{B,n}$

Before we finish this chapter we study some global properties of our constructed almost hyperkähler manifold and compare it to known results in literature. First we will calculate the homology groups and give an explicit description of the intersection form. Secondly we study possible compactifications and compare our results with the work of Chen & Chen (2019), Chen & Viaclovsky (2021) and Hein et al. (2021). Finally we compare the number of parameters in our construction to the dimension of the respective moduli spaces.

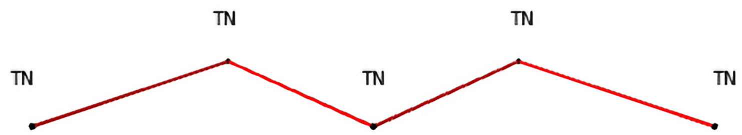



Figure 4: *The basespace of the multi-Taub-NUT space retracts to a line that connects all singularities.*

In order to study the topology of $M_{B,n}$, we first consider the topology of the multi-Taub-NUT space and we revisit the work by Sen (1997). Let $p_1, \ldots, p_n$ be an ordered list of points on $\mathbb{R}^3$ and consider the multi-Taub-NUT space which has $p_i$ as its singularities. As depicted in Figure 4, one can consider paths $\gamma_i$ that connect $p_i$ with $p_{i+1}$. By shrinking the base space, we retract the Taub-NUT space to a fiber bundle over this set of lines. Over each open line $\gamma_i$, the circle bundle must be trivial and by the Gibbons-Hawking metric, the circle radius of the fiber over each point is $\frac{2\pi\epsilon}{\sqrt{h_\epsilon}}$. From the construction of the harmonic function $h_\epsilon$, this circle radius must be finite and is zero only at the endpoints of the line segments. Hence the bundle over the line segment $\gamma_i$ is homotopic to a sphere and, as shown in Figure 5, these spheres only touch at the points $p_i$. Therefore, the multi-Taub-NUT space is homotopic to a chain of wedge sums of $n-1$ spheres.

By perturbing the paths $\gamma_i$ one can calculate the intersection matrix for the multi-Taub-NUT space. Because the endpoints of line segments are fixed, intersection can only happen at these points. According to Sen (1997), each sphere has self-intersection -2 and only intersects its neighbours with a factor $+1$. Hence the inter-

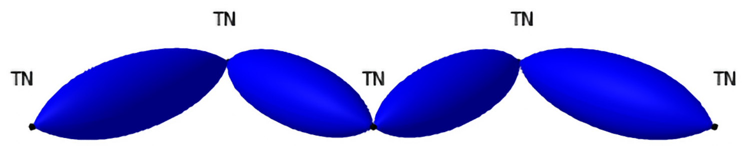



Figure 5: *The multi-Taub-NUT space retracts to a wedge sum of 2-spheres.*

section matrix is of the form

$$\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{pmatrix}$$

which is the negative Cartan matrix of a $A_{n-1}$-type Dynkin diagram.

In order to calculate the topology of $M_{B,n}$, we will use this retraction argument in combination with the Mayer-Vietoris sequence repeatedly. Namely, we will use the topology of $M_{\mathbb{R}^3,n}$ to calculate the topology of $M_{\mathbb{R}^2\times S^1,n}$, which will be used to calculate the topology of $M_{\mathbb{R}\times T^2,n}$. Using the explicit pictures like Figure 5, we calculate their intersection matrices.

**Proposition 3.24.** *When $B = \mathbb{R}^3$ and there are no non-fixed points $p_i$, the space $M_{B,n}$ retracts to $\mathbb{R}P^2$ and its homology is*

$$H_k(M_{\mathbb{R}^3,0}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}_2 & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

*When there are non-fixed points $p_i$, the homology of $M_{\mathbb{R}^3,n}$ is given by*

$$H_k(M_{\mathbb{R}^3,n}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}^n & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* First consider the case $n \neq 0$.

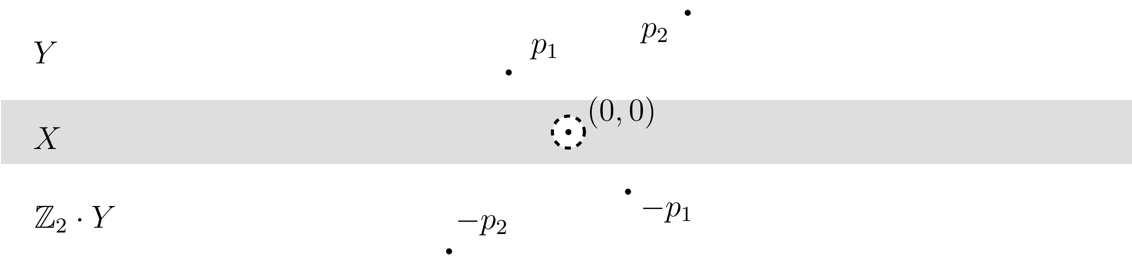



Figure 6: *The underlying manifold $M_{\mathbb{R}^3,n}$ can be seen as the union of the Atiyah-Hichin manifold and the Multi-Taub-NUT space.*

Let $\delta > 0$ be sufficiently small and rotate $\mathbb{R}^3$ such that the plate $X := (-\delta, \delta) \times \mathbb{R}^2 \subset B$ does not contain any of the non-fixed singularities $\pm p_i$. As shown in Figure 6, the base space $B'$ without the plate $X$ has two connected components, which can be identified using the antipodal map. Denote one of these connected components as $Y$. The antipodal map sends $X$ onto itself and therefore the bulk space $P/\mathbb{Z}_2$ can be written as

$$P/\mathbb{Z}_2 = (P|_X)/\mathbb{Z}_2 \cup P|_Y.$$

From the gluing construction we identify $M_{\mathbb{R}^3,n}$ with $\widetilde{P|_X/\mathbb{Z}_2} \cup \widetilde{P|_Y}$, where $\widetilde{P|_X/\mathbb{Z}_2}$ is the connected sum of $P|_X/\mathbb{Z}_2$ with the Atiyah-Hitchin manifold and $\widetilde{P|_Y}$ is the connected sum of $P|_Y$ with $n$ copies of Taub-NUT. The space $P|_X/\mathbb{Z}_2$ retracts to its boundary at the origin, which, after the connected sum construction, will be identified with the asymptotic region of the Atiyah-Hitchin manifold. Because the Atiyah-Hitchin manifold retracts to $\mathbb{R}P^2$, $\widetilde{P|_X/\mathbb{Z}_2}$, must also retract to $\mathbb{R}P^2$.

The space $\widetilde{P|_Y}$ can be studied in a similar manner as in Sen (1997), and hence it is homotopic to the wedge sum of $n-1$ copies of $S^2$. In order to apply the Mayer-Vietoris sequence we need to calculate $\widetilde{P|_X/\mathbb{Z}_2} \cap \widetilde{P|_Y}$. Because the two connected components $\{\pm\delta\} \times \mathbb{R}^2$ of the boundary of $X$ are identified by the antipodal map, $\widetilde{P|_X/\mathbb{Z}_2} \cap \widetilde{P|_Y}$ is diffeomorphic to a circle bundle over $\mathbb{R}^2$. Therefore, we get the following long exact sequence:

$$\ldots \longleftarrow \tilde{H}_k(M_{\mathbb{R}^3,n}) \longleftarrow \tilde{H}_k(\mathbb{R}P^2) \oplus \tilde{H}_k(\textstyle\bigvee_{i=1}^{n-1} S^2) \longleftarrow \tilde{H}_k(S^1) \longleftarrow \ldots$$

$$\begin{tikzcd}
0 & \tilde{H}_0(M_{\mathbb{R}^3,n}) \arrow[l] & 0 \oplus 0 \arrow[l] & 0 \arrow[l] & \\
& H_1(M_{\mathbb{R}^3,n}) & \mathbb{Z}_2 \oplus 0 \arrow[l] & \mathbb{Z} \arrow[l, "\partial"'] & \\
& H_2(M_{\mathbb{R}^3,n}) & 0 \oplus \mathbb{Z}^{n-1} \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

We directly conclude $H_0(M_{\mathbb{R}^3,n}) = \mathbb{Z}$ and $H_k(M_{\mathbb{R}^3,n}) = 0$ for $k > 2$. In order to calculate the remaining homology groups, we have to consider the map $\partial\colon H_1(S^2) \to H_1(\mathbb{R}P^2)$ in the long exact sequence. This map is the embedding of a fiber over a point into the $\widetilde{P|_X/\mathbb{Z}_2}$. As explained in Section 2.1, this fiber is homotopic to the generator of $H_1$ inside the Atiyah-Hitchin manifold and so $\partial(1) = [1]$. Using $H_1(M_{\mathbb{R}^3,n}) = \mathbb{Z}_2/\operatorname{im}(\partial)$ and $H_2(M_{\mathbb{R}^3,n}) = \mathbb{Z}^{n-1} \oplus \ker(\partial)$ we conclude

$$H_k(M_{\mathbb{R}^3,n}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}^n & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

When there are no non-fixed points, i.e. n=0, the manifold $M_{\mathbb{R}^3,n}$ retracts to $\widetilde{P|_X/\mathbb{Z}_2}$, which is homotopic to $\mathbb{R}P^2$. $\square$

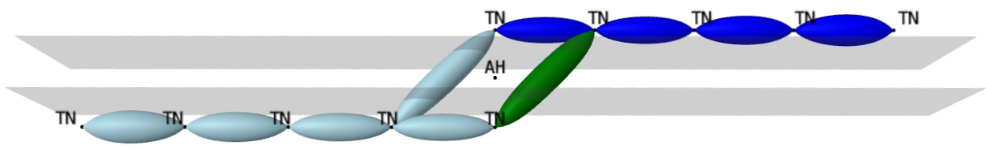



Figure 7: *Similarly to how Figure 5 depicts the 2-cycles inside the multi-Taub-NUT space, this figure depicts the 2-cycles with self-intersection -2 inside $P|_X \cup \widetilde{P|_Y} \cup \mathbb{Z}_2 \cdot \widetilde{P|_Y}$. The grey planes depict the boundary between these regions. The dark-blue spheres form a basis of $H_2(\widetilde{P|_Y})$ with $A_4$ as intersection matrix. The green sphere is the extra 2-cycle in $\ker\partial \subseteq H_2(M_{\mathbb{R}^3,5})$. The light-blue spheres are the $\mathbb{Z}_2$ images of the other spheres. The dark blue and green spheres form a basis of $H_2(M_{\mathbb{R}^3,5})$ such that its intersection matrix is the negative Cartan matrix of $D_5$.*

Sen (1997) argued that the intersection matrix for $M_{\mathbb{R}^3,n}$ is the negative Cartan matrix for a $D_n$ Dynkin diagram. His argument is as follows: As shown in Proposition

3.24, $H_2(M_{\mathbb{R}^3,n})$ is the direct sum of second homology group of the multi-Taub-NUT space with $n$ singularities with an extra copy of $\mathbb{Z}$. As explained at the beginning of this section, the multi-Taub-NUT space has a basis of 2-cycles such that the intersection matrix is given by the negative Cartan matrix of $A_{n-1}$. Again we can visualise this by retracting $Y$ to a set of line segments between the points $p_1, \ldots, p_n$ and considering the fiber bundle over these lines. In Figure 7, the fiber bundle over these line segments is depicted by dark blue spheres.

According to Proposition 3.24, there is one extra 2-cycle. From the Mayer-Vietoris sequence this 2-cycle must intersect the boundary between $\widetilde{P|_X/\mathbb{Z}_2}$ and $\widetilde{P|_Y}$ by two generators of $H_1(S^1)$. According to Sen (1997), we can find this extra 2-cycle by picking a path between $p_2$ and $\mathbb{Z}_2 \cdot p_1$ and considering the fiber bundle over this path. Using the same argument as before, this must be a 2-sphere with self intersection $-2$. In figure 7 this extra cycle is shown as the green sphere, and indeed this sphere intersects the boundary exactly twice. Because this extra 2-cycle intersects the sphere between $p_1$ and $p_2$ twice, but with opposite orientation, the intersection matrix is of the form

$$\begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & 1 \\ & & & & 1 & -2 & \\ & & & & 1 & & -2 \end{pmatrix}$$

This is the negative Cartan matrix of type $D_n$. This argument only works for $n \geq 4$. In Table 1 we calculate the intersection matrices for $n \leq 4$.

For the cases $2 \leq n \leq 4$, the intersection matrix for $M_{\mathbb{R}^3,n}$ is also given by the negative Cartan matrix of a $D_n$-Dynkin diagram. The case $M_{\mathbb{R}^3,1}$ does not fit into this framework. To find its self-intersection one notices that the generator for $M_{\mathbb{R}^3,1}$ is equivalent to the sum of the generators in $M_{\mathbb{R}^3,2}$ in Table 1. Because the intersection matrix for $M_{\mathbb{R}^3,2}$ is diagonal, the self-intersections add up.

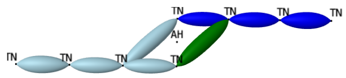
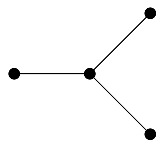
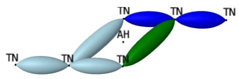
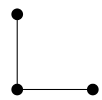
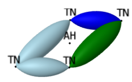
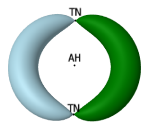
| $n$ | 2-cycles | Intersection matrix | Diagram | Type |
|---|---|---|---|---|
| 4 |  | $\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & 1 \\ & 1 & -2 & \\ & 1 & & -2 \end{pmatrix}$ |  | $D_4$ |
| 3 |  | $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}$ |  | $A_3$ |
| 2 |  | $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ |  | $A_1 + A_1$ |
| 1 |  | $(-4)$ | N/A | N/A. |

Table 1: *Intersection matrices for $M_{\mathbb{R}^3,n}$ with $n \leq 4$.*

**Proposition 3.25.** *When $B = \mathbb{R}^2 \times S^1$ and there are no non-fixed points $p_i$, the homology groups of $M_{\mathbb{R}^2\times S^1,0}$ are*

$$H_k(M_{\mathbb{R}^2\times S^1,0}) = \begin{cases} \mathbb{Z} & \text{if } k \in \{0,2\} \\ \mathbb{Z}_2 & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

*When there are non-fixed points $p_i$, the homology of $M_{\mathbb{R}^2\times S^1,n}$ is given by*

$$H_k(M_{\mathbb{R}^2\times S^1,n}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}^{n+1} & \text{if } k = 2 \\ 0 & \text{otherwise,} \end{cases}$$

*and the intersection matrix is given by the negative Cartan matrix of the extended Dynkin diagram of type $D_n$.*

*Proof.* Let $\delta > 0$ be sufficiently small. Consider $X := (\mathbb{R}^2 \times (-\delta,\delta)) \cap B' \subseteq \mathbb{R}^2 \times S^1$ and perturb $X$ such that it doesn't contain any of the non-fixed singularities $\pm p_i$. Denote $Y$ as the complement of $X$ inside $B'$. By construction both are $\mathbb{Z}_2$ invariant under the antipodal map, and hence the bulk space $P/\mathbb{Z}_2$ can be written as

$$P/\mathbb{Z}_2 = P|_X/\mathbb{Z}_2 \cup P|_Y/\mathbb{Z}_2.$$

From the gluing construction we identify $M_{\mathbb{R}^2\times S^1,n} = \widetilde{P|_X/\mathbb{Z}_2} \cup \widetilde{P|_Y/\mathbb{Z}_2}$, where $\widetilde{P|_X/\mathbb{Z}_2}$ and $\widetilde{P|_Y/\mathbb{Z}_2}$ are completions of $P|_X/\mathbb{Z}_2$ and $P|_Y/\mathbb{Z}_2$ respectively using Atiyah-Hitchin manifolds and Taub-NUT spaces. Using a similar construction as in Proposition 3.24, we identify the topology of $\widetilde{P|_X/\mathbb{Z}_2}$ and $\widetilde{P|_Y/\mathbb{Z}_2}$ with the topology of $M_{\mathbb{R}^3,0}$ and $M_{\mathbb{R}^3,n}$ respectively. The intersection $\widetilde{P|_X/\mathbb{Z}_2} \cap \widetilde{P|_Y/\mathbb{Z}_2}$ must be an $S^1$-bundle over a plane, which retracts to a circle. With this information we fill in the Mayer-Vietoris sequence:

$$\dots \longleftarrow \tilde{H}_k(M_{\mathbb{R}^2\times S^1,n}) \longleftarrow \tilde{H}_k(M_{\mathbb{R}^3,0}) \oplus \tilde{H}_k(M_{\mathbb{R}^3,n}) \longleftarrow \tilde{H}_k(S^1) \longleftarrow \dots$$

$$\begin{tikzcd}
0 & \tilde{H}_0(M_{\mathbb{R}^2\times S^1,n}) \arrow[l] & 0\oplus 0 \arrow[l] & 0 \arrow[l] \\
& H_1(M_{\mathbb{R}^2\times S^1,n}) \arrow[ur, to=1-4] & \mathbb{Z}_2 \oplus \delta_n\mathbb{Z}_2 \arrow[l] & \mathbb{Z} \arrow[l, "\partial"'] \\
& H_2(M_{\mathbb{R}^2\times S^1,n}) \arrow[to=2-4] & 0 \oplus \mathbb{Z}^n \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Here, we used $\delta_n\mathbb{Z}_2$ as a shorthand for $H_1(M_{R^3,n}) = \begin{cases} \mathbb{Z}_2 & \text{if } n=0 \\ 0 & \text{if } n\neq 0. \end{cases}$ The Mayer-Vietoris sequence implies

$$H_k(M_{\mathbb{R}^2\times S^1,n}) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ \frac{\mathbb{Z}_2\oplus\delta_n\mathbb{Z}_2}{\operatorname{im}\partial} & \text{if } k=1 \\ \mathbb{Z}^n \oplus \ker\partial & \text{if } k=2 \\ 0 & \text{otherwise.} \end{cases}$$

In order to give $H_k(M_{\mathbb{R}^2\times S^1,n})$ explicitly, we need to study the map $\partial\colon \mathbb{Z} \to \mathbb{Z}_2\oplus\delta_n\mathbb{Z}_2$. Just as in Proposition 3.24, $\partial(1)$ embeds a circle fiber over a point into $\widetilde{P|_X/\mathbb{Z}_2}$ and $\widetilde{P|_Y/\mathbb{Z}_2}$. As discussed before, this circle fiber maps to the generator of the $\mathbb{R}P^2$ in the Atiyah-Hitchin manifold. We conclude that

$$\partial(1) = \begin{cases} ([1],[1]) & \text{if } n = 0 \\ [1] & \text{if } n \neq 0, \end{cases}$$

and therefore

$$H_k(M_{\mathbb{R}^2\times S^1,n}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \delta_n \mathbb{Z}_2 & \text{if } k = 1 \\ \mathbb{Z}^{n+1} & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Using Sen's method, one can construct the generators of $H_2(M_{\mathbb{R}^2\times S^1,n})$ and calculate their intersection matrix. In table 2 these generators are explicitly given and their intersection matrix correspond to the negative Cartan matrix related to the extended Dynkin diagram of type $D_n$. □

*Remark* 3.26. There is an alternative way to visualise the extra 2-cycle inside $M_{\mathbb{R}^2\times S^1,n}$. Indeed, consider the loop $\{\pm x\} \times S^1 \subseteq \mathbb{R}^2 \times S^1$ for some $|x| \ggg 1$. In the $\mathbb{Z}_2$ quotient this maps to a single loop and hence the fiber bundle over this loop will be a 2-torus inside $M_{\mathbb{R}^2\times S^1,n}$. This loop intersects the boundary $X/\mathbb{Z}_2 \cap Y/\mathbb{Z}_2$ exactly twice and hence the torus intersects $\widetilde{P|_X/\mathbb{Z}_2} \cap \widetilde{P|_Y/\mathbb{Z}_2}$ at two copies of $S^1$. One can check that these two copies of $S^1$ have the same orientation and hence they lie in the kernel of $\partial\colon H_1(S^1) \to H_1(M_{\mathbb{R}^3,0}) \oplus H_1(M_{\mathbb{R}^3,n})$. Because $H_2(M_{\mathbb{R}^2\times S^1,n}) = \mathbb{Z}^n \oplus \ker\partial$, this torus can be viewed as one of the generators of $H_2$. By varying $|x|$ one sees that this torus has no self-intersection and in this new basis the intersection matrix is

$$\begin{pmatrix} D_n & 0 \\ 0 & 0 \end{pmatrix}.$$

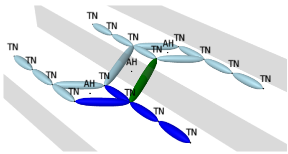
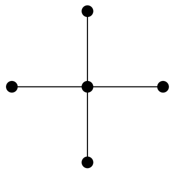
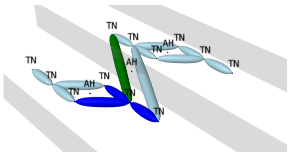
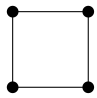
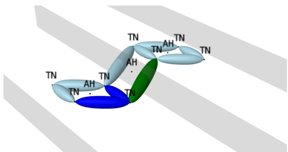
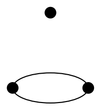
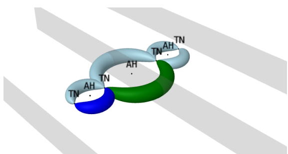
| $n$ | 2-cycles | Intersection matrix | Diagram | Type |
|---|---|---|---|---|
| 4 |  | $\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & 1 & 1 \\ & 1 & -2 & & \\ & 1 & & -2 & \\ & 1 & & & -2 \end{pmatrix}$ |  | $\tilde{D}_4$ |
| 3 |  | $\begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix}$ |  | $\tilde{A}_3$ |
| 2 |  | $\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}$ |  | $A_1 + \tilde{A}_1$ |
| 1 |  | $\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}$ | N/A | N/A. |

Table 2: *Intersection matrices for $M_{\mathbb{R}^2\times S^1,n}$. These generators are constructed by restricting the fiber bundle over paths between the non-fixed points $p_i$ inside the universal cover of $B = \mathbb{R}^2 \times S^1$. The dark blue spheres are the generators of $H_2(M_{R^n,n})$ given in Table 1. The green spheres is the extra 2-cycle that are induced by the kernel of $\partial\colon H_1(S^1) \to H_1(M_{\mathbb{R}^3,0}) \oplus H_1(M_{\mathbb{R}^3,n})$. The light blue spheres are the images of the dark blue and green spheres under the antipodal map and the action of $\mathbb{Z}$. The gray planes depict the boundary of the fundamental domain of $\mathbb{R}^2 \times S^1$ inside its universal cover.*

**Proposition 3.27.** *When $B = \mathbb{R} \times T^2$ and there are no non-fixed points $p_i$, the homology groups of $M_{\mathbb{R}\times T^2,0}$ are*

$$H_k(M_{\mathbb{R}\times T^2,0}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}_2 & \text{if } k = 1 \\ \mathbb{Z}^3 & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

*When there are non-fixed points $p_i$, the homology of $M_{\mathbb{R}\times T^2,n}$ is given by*

$$H_k(M_{\mathbb{R}\times T^2,n}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}^{n+3} & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let $\delta > 0$ be sufficiently small. Consider $X := (\mathbb{R}\times S^1 \times (-\delta, \delta)) \cap B' \subseteq \mathbb{R}\times T^2$ and perturb $X$ such that it doesn't contain any of the non-fixed singularities $\pm p_i$. Denote $Y$ as the complement of $X$ inside $B'$. By construction both are $\mathbb{Z}_2$ invariant under the antipodal map and hence the bulk space $P/\mathbb{Z}_2$ can be written as

$$P/\mathbb{Z}_2 = P|_X/\mathbb{Z}_2 \cup P|_Y/\mathbb{Z}_2.$$

From the gluing construction we identify $M_{\mathbb{R}\times T^2,n} = \widetilde{P|_X/\mathbb{Z}_2} \cup \widetilde{P|_Y/\mathbb{Z}_2}$, where $\widetilde{P|_X/\mathbb{Z}_2}$ and $\widetilde{P|_Y/\mathbb{Z}_2}$ are completions of $P|_X/\mathbb{Z}_2$ and $P|_Y/\mathbb{Z}_2$ respectively using Atiyah-Hitchin manifolds and Taub-NUT spaces. Using a similar construction as in Proposition 3.25, we identify the topology of $\widetilde{P|_X/\mathbb{Z}_2}$ and $\widetilde{P|_Y/\mathbb{Z}_2}$ with the topology of $M_{\mathbb{R}^2\times S^1,0}$ and $M_{\mathbb{R}^2\times S^1,n}$ respectively. The intersection $\widetilde{P|_X/\mathbb{Z}_2} \cap \widetilde{P|_Y/\mathbb{Z}_2}$ must be an $S^1$-bundle over $\mathbb{R}\times S^1 \times \{\delta\}$, which is homotopic to $T^2$. This gives us the following long exact sequence:

$$\cdots \longleftarrow \tilde{H}_k(M_{\mathbb{R}\times T^2,n}) \longleftarrow \tilde{H}_k(M_{\mathbb{R}^2\times S^1,0}) \oplus \tilde{H}_k(M_{\mathbb{R}^2\times S^1,n}) \longleftarrow \tilde{H}_k(T^2) \longleftarrow \cdots$$

$$\begin{tikzcd}
0 & \tilde{H}_0(M_{\mathbb{R}\times T^2,n}) \arrow[l] & 0 \oplus 0 \arrow[l] & 0 \arrow[l] & \\
 & H_1(M_{\mathbb{R}\times T^2,n}) \arrow[urr] & \mathbb{Z}_2 \oplus \delta_n\mathbb{Z}_2 \arrow[l] & \mathbb{Z}^2 \arrow[l, "\partial_1"'] & \\
 & H_2(M_{\mathbb{R}\times T^2,n}) \arrow[urr] & \mathbb{Z} \oplus \mathbb{Z}^{n+1} \arrow[l] & \mathbb{Z} \arrow[l, "\partial_2"'] & \\
 & H_3(M_{\mathbb{R}\times T^2,n}) \arrow[urr] & 0 \oplus 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

We conclude

$$H_k(M_{\mathbb{R}\times T^2,n}) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ \frac{\mathbb{Z}_2 \oplus \delta_n \mathbb{Z}_2}{\operatorname{im}\partial_1} & \text{if } k=1 \\ \ker\partial_1 \oplus \frac{\mathbb{Z}\oplus\mathbb{Z}^{n+1}}{\operatorname{im}\partial_2} & \text{if } k=2 \\ \ker\partial_2 & \text{if } k=3 \\ 0 & \text{otherwise.} \end{cases}$$

In order to give $H_k(M_{\mathbb{R}\times T^2,n})$ explicitly, we need to study the map $\partial_i\colon H_i(T^2) \to H_i(M_{\mathbb{R}^2\times S^1,0}) \oplus H_i(M_{\mathbb{R}^2\times S^1,n})$. The generator of $H_2(T^2)$ can be viewed as one of the tori at infinity, and hence by Remark 3.26, the map $\partial_2$ is injective. Therefore,

$$H_k(M_{\mathbb{R}\times T^2,n}) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ \frac{\mathbb{Z}_2 \oplus \delta_n \mathbb{Z}_2}{\operatorname{im}\partial_1} & \text{if } k=1 \\ \ker\partial_1 \oplus \mathbb{Z}^{n+1} & \text{if } k=2 \\ 0 & \text{otherwise.} \end{cases}$$

Secondly, recall that $T^2$ was the $S^1$-bundle over $\mathbb{R}\times S^1\times\{\delta\}$ and so $H_1(T^2)$ is generated by a fiber over a point and the non-trivial circle in the base space. Similarly, as in Propositions 3.24 and Propositions 3.25, the map $\partial_1$ maps the fiber over a point to the generator of $\mathbb{Z}_2$ inside the Atiyah-Hitchin manifold. The circle in the base space maps to a contractible loop inside the Atiyah-Hitchin manifold and hence

$$\partial_1([1]_{fiber}) = \begin{cases} ([1],[1]) & \text{if } n=0 \\ [1] & \text{if } n\neq 0, \end{cases} \qquad \text{and} \qquad \partial_1([1]_{base}) = 0.$$

This concludes

$$H_k(M_{\mathbb{R}\times T^2,n}) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ \delta_n\mathbb{Z}_2 & \text{if } k=1 \\ \mathbb{Z}^{n+3} & \text{if } k=2 \\ 0 & \text{otherwise.} \end{cases}$$

□

*Remark* 3.28. Similarly, as in Remark 3.26, we can identify the tori at infinity. In this case, there are two tori on the circle bundle at infinity and both can be visualised as the circle bundle over the 1-cycles on the asymptotic bases pace. Indeed, we already found one of these tori, which can be viewed as one of the generators of $\mathbb{Z}^{n+1}$. By studying $\ker\partial_1$ in a similar manner as in Remark 3.26, we can conclude that the other torus must be a generator or $\ker\partial_1 \simeq \mathbb{Z}^2$.

**The moduli space**

We compare $M_{B,n}$ with the known classifications of gravitational instantons. All gravitational instantons are classified by Sun & Zhang (2021) by comparing them to model metrics at infinity. The list of all possible model metrics is given in Section 6.4 in their paper. For the case $B=\mathbb{R}^3$, recall from Lemma 3.4 that the metric on $P$ approximates the Gibbons-Hawking metric on $\mathbb{R}^3\setminus\{0\}$ with the harmonic function $h = \text{const} + \frac{2n-4}{2r}$. After completing $P/\mathbb{Z}_2$ using the gluing construction and perturbing the metric, we still expect that $M_{\mathbb{R}^3,n}$ approximates the same model metric at infinity. Therefore, the space $M_{\mathbb{R}^3,n}$ is an example of an ALF-$D_n$ gravitational instanton[12]. The name ALF-$D_k$ is not arbitrary: According to Chen & Chen (2019) Remark 6.3, gravitational instantons of type ALF-$D_n$ have an intersection matrix is related to the $D_n$ Dynkin diagram. This is exactly what Sen (1997) found.

Chen & Chen (2021a) found a Torelli theorem for ALF-type gravitational instantons: Up to triholomorphic isometries all ALF-type gravitational instantons can be uniquely classified by their model at infinity and their periods. For ALF spaces the model at infinity is fully determined by the degree of the circle bundle at infinity and the length of its fiber at infinity. These parameters correspond to the number of non-fixed singularities $p_i$ and $\epsilon$ respectively. To calculate the period we have to integrate the hyperkähler triple over a basis of $H_2(M_{\mathbb{R}^3,n})$ where each element has

[12] According to Definition 6.8 in Sun & Zhang (2021).

self-intersection $-2$. There are 3 Kähler triples and the dimension of $H_2(M_{\mathbb{R}^3,n})$ is $n$. Hence, the moduli space of ALF metrics with a fixed model space is $3n$. This number corresponds with the $n$ possible positions of the nuts in $\mathbb{R}^3$.

When $B = \mathbb{R}^2 \times S^1$, the asymptotic metric of $(P/\mathbb{Z}_2, g^{GH})$ is given by

$$g^{GH} \sim \begin{cases} \epsilon(8-2n)\log r \cdot (g_{\mathbb{R}^2/\mathbb{Z}_2} + g_{S^1}) + \frac{\epsilon}{(8-2n)\log r}\eta_\infty^2 & \text{if } 0 \le n < 4 \\ g_{\mathbb{R}^2/\mathbb{Z}_2} + g_{S^1} + \epsilon^2(\mathrm{d}\, t + b\, \mathrm{d}\, z)^2 & \text{if } n = 4. \end{cases} \tag{9}$$

For the case $n < 4$, gravitational instantons with these asymptotics are called[13] ALG*-$I^*_{4-n}$. For the case $n = 4$, it is called[14] ALG$_{\frac{1}{2}}$, because $g^{GH}$ approximates the metric of a flat torus bundle over the 2-dimensional cone with angle $\frac{1}{2} \cdot 2\pi$.

In order to explain the name, we have to deviate to the study of rational elliptic surfaces: According to Chen & Viaclovsky (2021) Theorem 1.5 and Chen & Chen (2021b) Theorem 1.2 gravitational instantons of type ALG*/ALG can be compactified to a rational elliptic surface by adding a singular Kodaira fiber. In order to find the type of fiber, we first need to understand the elliptic fibration of $P/\mathbb{Z}_2$ at infinity. For this we equip the plane $\mathbb{R}^2$ inside $B = \mathbb{R}^2 \times S^1$ with the standard complex structure and on the asymptotic region of $P/\mathbb{Z}_2$ we consider the following composition of maps

$$P/\mathbb{Z}_2 \to (\mathbb{R}^2 \times S^1)/\mathbb{Z}_2 \xrightarrow{\simeq} (\mathbb{C} \times S^1)/\mathbb{Z}_2 \to \mathbb{C}/\mathbb{Z}_2.$$

This composition map gives $P/\mathbb{Z}_2$ the structure of a torus fibration at infinity. Indeed, using the coordinates $\{r, \phi, z, t\}$ on $P$ from Equation 8, one can check that this projection map is holomorphic with respect to the complex structure $\omega_z = \epsilon\, \mathrm{d}\, z \wedge \eta + h_\epsilon *^B \mathrm{d}\, z$. The complex structure $I_z$ corresponding to $\omega_z$ has the property $I_z\, \mathrm{d}\, z = -\frac{\epsilon}{h_\epsilon}\eta$, which implies the bi-holomophicity of the transition functions.

Knowing that $P/\mathbb{Z}_2$ has an elliptic fibration, we can find the singular Kodaira fiber by calculating the monodromy at infinity. For this we use the coordinates from Equation 8 and we consider the path $\gamma_{r_0,\phi_0,z_0,t_0}(s) = (r_0, \phi_0 + s, z_0, t_0 + c\, z_0\, s)$, which

[13] e.g. Definition 6.12 in Sun & Zhang (2021).
[14] e.g. Definition 6.11 in Sun & Zhang (2021).

is horizontal with respect to the asymptotic connection[15] $\eta_\infty = \mathrm{d}\, t - cz\, \mathrm{d}\, \phi + b\, \mathrm{d}\, z$. In these coordinates, the $\mathbb{Z}_2$ action on $P$ is given by

$$\mathbb{Z}_2 \cdot (r, \phi, z, t) = (r, \phi + \pi, -z, -t)$$

and so $\gamma_{r_0,\phi_0,z_0,t_0}(\pi)$ is equivalent to

$$\gamma_{r_0,\phi_0,z_0,t_0}(\pi) = (r_0, \phi_0 + \pi, z_0, t_0 + c\, z_0\, \pi) \sim (r_0, \phi, -z_0, -t_0 - c\, z_0\, \pi).$$

Hence the generator $[1_t] \in H_1(T^2)$, that is generated by the path along the circle fiber, maps to $-[1_t]$ under the monodromy along $\gamma$. Similarly, the other generator $[1_z] \in H_1(T^2)$ maps to $-[1_z] - \frac{c}{2}[1_t]$. When $B = \mathbb{R}^2 \times S^1$, the first Chern class $c$ equals $8 - 2n$, and hence the monodromy at infinity is given by

$$\begin{pmatrix} -1 & 4-n \\ 0 & -1 \end{pmatrix}.$$

Under the Kodaira classification $M_{\mathbb{R}^2 \times S^1, n}$ can be compactified by an $I^*_{4-n}$ fiber, which explains the name.

In Chen & Viaclovsky (2021), there is a Torelli Theorem for ALG* gravitational instantons: Up to triholomorphic isometries all ALG*-type gravitational instantons can be uniquely identified by their model at infinity and their periods. The model at infinity is determined by the lattice and a global scale. Up to rotation, a one-dimensional lattice is only determined by the length of its generator and so the model at infinity is determined by two parameters.

In their paper they argue that the period map overcounts the moduli space. One way to see this is by integrating the Kähler forms over the torus at infinity. Here the Kähler forms are fixed by the model at infinity and so these integrals can only reveal information of this model. Therefore, they claim that the dimension of the moduli space of ALG* gravitational instantons with fixed model at infinity is $3(\beta_2 - 1)$. Using Proposition 3.25 we see that the dimension is $3n$. Again this corresponds to the $n$ possible positions of the non-fixed singularities in $\mathbb{R}^2 \times S^1$.

Chen & Viaclovsky (2021) also found a Torelli theorem for ALG-type gravitational

[15] This connection is given in Proposition 3.13.

instantons. As seen in Equation 9, the model metric is determined by three parameters, the length of the circle in the base space (i.e. $g_{S^1}$), the size of the circle fiber (i.e. $\epsilon$) and the choice of model connection (i.e. $b$). With these choices fixed, they argue that the dimension of the moduli space is $3(\beta_2 - 1) = 12$. Again we expect this as we have 12 degrees of freedom in choosing the location of the nuts.

Finally we consider the $B = \mathbb{R} \times T^2$ case. In this situation, the asymptotic metric of $(P/\mathbb{Z}_2, g^{GH})$ is given by

$$g^{GH} \sim \begin{cases} \epsilon(16-2n)r \cdot (g_{\mathbb{R}^+} + g_{T^2}) + \frac{\epsilon}{(16-2n)r}\eta_\infty^2 & \text{if } 0 \leq n < 8 \\ g_{\mathbb{R}^+} + g_{T^2} + \epsilon^2(\mathrm{d}\,t + a\,\mathrm{d}\,y + b\,\mathrm{d}\,z)^2 & \text{if } n = 8. \end{cases}$$

For the case $n < 8$, gravitational instantons with these asymptotics are called[16] ALH*$_{8-n}$. For the case $n = 8$, it is called[17] ALH, because $g^{GH}$ approximates the product metric of $\mathbb{R}^+$ and a flat 3-dimensional torus.

Similarly to the ALG*/ALG case, gravitational instantons with ALH*/ALH ends can be compactified to rational elliptic surfaces by adding a (singular) Kodaira fiber. According to Chen & Chen (2021b) Theorem 1.2, ALH gravitational instantons can be compacified by the smooth $I_0$ fiber, and according to T. C. Collins et al. (2020) Theorem 1.4, ALH*$_k$ manifolds can be compactified by the non-regular $I_k$ fiber. In Remark 6.21, Sun & Zhang (2021) showed that $k$ must be between 1 and 9. By calculating the monodromy at infinity one can determine the type of fiber. The method is very similar to the one for the ALG*/ALG case and was explicitly done in Section 2.2 of Hein et al. (2022). The monodromy for this case is

$$\begin{pmatrix} 1 & 8-n \\ 0 & 1 \end{pmatrix}.$$

Hence one can compactify $M_{\mathbb{R}\times T^2,n}$ by adding an $I_{8-n}$ fiber.

T. Collins et al. (2022) and Lee & Lin (2022) determined the Torelli theorem for ALH* gravitational instantons. Up to triholomorphic isometries, all ALH*-type gravitational instantons can be uniquely identified by their model at infinity and the periods. The model at infinity is determined by the lattice and a global scale.

[16]e.g. Definition 6.16 in Sun & Zhang (2021).
[17]e.g. Definition 6.15 in Sun & Zhang (2021).

Up to rotation a 2-dimensional lattice is only determined the length its generators and their angle. With $3n$ degrees of freedom in choosing the location of the non-fixed singularities, we expect the total dimension of the moduli space to be $3n+7$. In a recent survey paper by T. C. Collins & Lin (2022), the dimension of the moduli space of Tian-Yau metrics was calculated and it agrees with our dimension count.

As explained in Chapter 2, there are two known constructions for ALH* gravitational instantons, due to Hein (2010) and Tian & Yau (1990). By T. C. Collins et al. (2020), there are ALH* gravitational instantons that are not generated by Hein's construction, but according to Hein et al. (2021) any ALH* gravitational instanton arises from the generalized Tian-Yau construction on some weak del Pezzo surface minus a smooth anticanonical divisor. Up to diffeomorphism[18] there are 10 different (weak) del Pezzo surfaces: $\mathbb{C}P^2$, the blow-up of $\mathbb{C}P^2$ at up to 8 points and $S^2 \times S^2$. The degree of the anti-canonical divisor is $9-k$ for $\mathrm{Bl}_k\,\mathbb{C}P^2$ and 8 for $S^2 \times S^2$. Given this degree, one can compare the construction by Tian-Yau and Hein: According to T. C. Collins et al. (2021), a del Pezzo surface without a smooth anticanonical divisor $D$ with $D^2 = d$ can be compactified to a rational elliptic surface by adding an $I_d$ fiber after performing a hyperkähler rotation. As the monodromy never allows us to glue in an $I_9$ fiber we conclude

**Proposition 3.29.** *For $0 \leq n \leq 8$, the space $M_{\mathbb{R}\times T^2,n}$ is not diffeomorphic to the complement of a smooth anticanonical divisor of $\mathbb{C}P^2$*

When $1 \leq n < 8$, the monodromy of $M_{\mathbb{R}\times T^2,n}$ suggests one can compactify it to a rational elliptic surface by adding an $I_{8-n}$ fiber. For the degrees one to seven, the del Pezzo surfaces are unique and so we have:

**Proposition 3.30.** *For $1 \leq n < 8$, the space $M_{\mathbb{R}\times T^2,n}$ is diffeomorphic to the complement of a smooth anticanonical divisor of the blowup of $\mathbb{C}P^2$ at $8-n$ points.*

Up to diffeomorphism, there are two del Pezzo surfaces of degree 8, namely $S^2 \times S^2$ and $\mathrm{Bl}_1\,\mathbb{C}P^2$. We claim that $\mathrm{Bl}_1\,\mathbb{C}P^2$ cannot be used to construct $M_{\mathbb{R}\times T^2,0}$.

[18] There are actually 11 different weak del Pezzo surfaces(Hidaka & Watanabe (1981), Theorem 3.4), but $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the second Hirzebruch surface are diffeomorphic.

**Proposition 3.31.** *The space $M_{\mathbb{R}\times T^2,0}$ is not diffeomorphic to the complement of a smooth anticanonical divisor of the blowup of $\mathbb{C}P^2$ at one point.*

*Proof.* Suppose it does arise from the Tian-Yau construction. Then $M_{\mathbb{R}\times T^2,0}$ can be compactified by gluing the disk bundle $D$ at infinity. The boundary $\partial D$ is an $S^1$-bundle over $T^2$ of degree 8. These identifications yields the following Mayer-Vietoris sequence:

$$\ldots \longleftarrow \tilde{H}_k(\mathrm{Bl}_1\,\mathbb{C}P^2) \longleftarrow \tilde{H}_k(D) \oplus \tilde{H}_k(M_{\mathbb{R}\times T^2,0}) \longleftarrow \tilde{H}_k(\partial D) \longleftarrow \ldots$$

$$\begin{tikzcd}
0 & 0 \arrow[l] & 0 \oplus 0 \arrow[l] & 0 \arrow[l] & \\
 & 0 \arrow[urr] & \mathbb{Z}^2 \oplus \mathbb{Z}_2 \arrow[l] & \mathbb{Z}^2 \oplus \mathbb{Z}_8 \arrow[l, "\alpha"'] & \\
 & \mathbb{Z}^2 \arrow[urr, "\beta"'] & \mathbb{Z} \oplus \mathbb{Z}^3 \arrow[l, "\gamma"'] & \mathbb{Z}^2 \arrow[l, "\delta"'] & \\
 & 0 \arrow[urr] & 0 \oplus 0 \arrow[l] & \mathbb{Z} \arrow[l] & \\
 & \mathbb{Z} \arrow[urr] & 0 \oplus 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Our goal is to show that the short exact sequence

$$0 \longrightarrow \ker\beta \longrightarrow H_2(\mathrm{Bl}_1\,\mathbb{C}P^2) \xrightarrow{\beta} \mathrm{im}\,\beta \longrightarrow 0$$

cannot exist. First we study $\mathrm{im}\,\beta = \ker\alpha$. From the Gysin sequence it follows that the free part of $H_1(\partial D)$ is generated by the homology of the base space of $D$. Therefore the map $\alpha$ is of the form

$$\begin{aligned}
\mathbb{Z}^2 \oplus \mathbb{Z}_8 &\xrightarrow{\alpha} \mathbb{Z}^2 \oplus \mathbb{Z}_2 \\
(1,0,0) &\longmapsto (1,0,\ldots) \\
(0,1,0) &\longmapsto (0,1,\ldots)
\end{aligned}$$

Because $\alpha$ must be surjective,

$$\begin{aligned}
\mathbb{Z}^2 \oplus \mathbb{Z}_8 &\xrightarrow{\alpha} \mathbb{Z}^2 \oplus \mathbb{Z}_2 \\
(0,0,1) &\longmapsto (0,0,1)
\end{aligned}$$

and hence the kernel of $\alpha$ must be isomorphic to $\mathbb{Z}_4$. We conclude

$$0 \longrightarrow \ker\beta \longrightarrow H_2(\mathrm{Bl}_1\,\mathbb{C}P^2) \xrightarrow{\beta} \mathbb{Z}_4 \longrightarrow 0.$$

Secondly, we study $\ker\beta = \operatorname{im}\gamma$. By the first isomorphism theorem, $\operatorname{im}\gamma = \frac{H_2(D)\oplus H_2(M_{\mathbb{R}\times T^2,0})}{\operatorname{im}\delta}$. According to Remark 3.28, we have $H_2(M_{\mathbb{R}\times T^2,n}) \simeq \mathbb{Z}^{n+1} \oplus H_2(\partial D)$ by identifying the tori at infinity with the generators of $H_2(\partial D)$. Hence, $\delta$ is the inclusion map and $\ker\beta = \operatorname{im}\gamma = H_2(D)\oplus\mathbb{Z} = \mathbb{Z}^2$.

We explicitly give the generators of $\ker\beta$: The first generator is the smooth anti-canonical divisor of $\mathrm{Bl}_1\,\mathbb{C}P^2$ to which $D$ retracts. We denote this divisor as $K^{-1}$. The other generator is the generator of $H_2(M_{\mathbb{R}\times T^2,0})$ that cannot be represented as a torus at infinity. We denote this 2-cycle as $C$. Both $K^{-1}$ and $C$ can be identified as 2-cycles on the interior of their domain, and hence $K^{-1}\cdot C = 0$. In summary,

$$\begin{array}{ccccccccc} & & (\ker\beta) & & & & (\operatorname{im}\beta) & & \\ 0 & \longrightarrow & \langle K^{-1}, C\rangle & \longrightarrow & H_2(\mathrm{Bl}_1\,\mathbb{C}P^2) & \xrightarrow{\beta} & \mathbb{Z}_4 & \longrightarrow & 0. \end{array}$$

Finally, we study $H_2(\mathrm{Bl}_1\,\mathbb{C}P^2)$. The space $H_2(\mathrm{Bl}_1\,\mathbb{C}P^2)$ has two generators. One is the generator of $H_2(\mathbb{C}P^2)$, which we denote by $H$. The other is the exceptional divisor $E$ from the blowup. They intersect as follows:

$$H\cdot H = 1 \qquad H\cdot E = 0 \qquad E\cdot E = -1.$$

In terms of $H$ and $E$, the anti-canonical divisor $K^{-1}$ is given by $K^{-1} = 3H - E$. As a sanity check, one can verify that this is the only 2-cycle up to orientation such that $K^2 = 8$. From the map $\beta$ we get the following short exact sequence:

$$\begin{array}{ccccccccc} & & (\ker\beta) & & (H_2(\mathrm{Bl}_1\,\mathbb{C}P^2)) & & (\operatorname{im}\beta) & & \\ 0 & \longrightarrow & \langle K^{-1}, C\rangle & \longrightarrow & \langle H, E\rangle & \xrightarrow{\beta} & \mathbb{Z}_4 & \longrightarrow & 0 \end{array}$$

We claim there is no $C \in \ker\beta$ which makes this sequence exact. Indeed, suppose $C = \alpha H + \beta E$ for some $\alpha,\beta\in\mathbb{Z}$. Because $K^{-1}$ and $C$ do not intersect,

$$K^{-1}\cdot C = (3H - E)\cdot(\alpha H + \beta E) = 3\alpha + \beta = 0$$

and hence $C = \alpha(H - 3E)$. At the same time, $E$ can be written as a $\mathbb{Z}$-linear combination of $H$ and $K^{-1}$. Therefore, $H_2(\mathrm{Bl}_1 \mathbb{C}P^2)$ is generated by $H$ and $K^{-1}$ and

$$C = \alpha(H - 3E) = \alpha(H - 3(3H - K^{-1})) = -8\alpha H + 3K^{-1}.$$

We conclude $\operatorname{im}\beta = \frac{\langle H,K^{-1}\rangle}{\langle C,K^{-1}\rangle} \simeq \mathbb{Z}_{8\alpha}$. There is no integer value for $\alpha$ which makes this equal to $\mathbb{Z}_4$. □

Alternatively, one can calculate the fundamental group of the complement of a smooth anti-canonical divisor of the blowup of $\mathbb{C}P^2$ at one point using the van Kampen theorem. This way one can show that this complement is simply connected and hence differs from $M_{\mathbb{R}\times T^2,0}$.

# 4 Fredholm theory for AL$\mathfrak{X}$ gravitational instantons

In this chapter we will study the Laplacian on the asymptotic geometry of $P/\mathbb{Z}_2$. A standard tool used in the literature is the use of weighted spaces. For example, Bartnik (1986) considers the norm

$$\| \dots \|^2_{W^{k,2}_\delta(\mathbb{R}^n \setminus K)} := \sum_{j=0}^{k} \| r^{j-\delta-\frac{n}{2}} \nabla^j \dots \|^2_{L^2(\mathbb{R}^n \setminus K)}$$

on $\mathbb{R}^n \setminus K$ for all $\delta \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$ and some large compact set $K$. By introducing $r^{-\delta-\frac{n}{2}}$, one forces a certain polynomial decay rate at infinity. Because the harmonic functions on $\mathbb{R}^n$ are polynomials for $n \geq 3$, one can dictate the dimension of the (co)-kernel by choosing $\delta$. When $\delta$ lies in the range $(2-n, 0)$, $\Delta$ is bijective.

Instead of using weighted norms and a fixed operator, one can consider the weighted operator $L_\delta := r^{2-\delta} \Delta^{Eucl}(r^\delta \dots)$ on the fixed Banach space induced by the metric $g_{cf} := r^{-2} g_{Eucl}$. It turns out that these methods are equivalent. Indeed, because $\mathrm{d}\log r$ and $\nabla^g \mathrm{d}\log r$ are uniformly bounded in $g_{cf}$, there exist some constants $c, C > 0$ such that for any $u \in W^{2,2}_\delta(\mathbb{R}^n \setminus K)$,

$$c\|u\|_{W^{2,2}_\delta(\mathbb{R}\setminus K)} \leq \|r^{-\delta} u\|_{W^{2,2}_{cf}(\mathbb{R}\setminus K)} \leq C\|u\|_{W^{2,2}_\delta(\mathbb{R}\setminus K)}.$$

Moreover, the operator $L_\delta$ is strictly elliptic with respect to $g_{cf}$ and so for any pair of bounded open balls $U \subset\subset U'$ not containing the origin, the Schauder estimate by Evans (1998)

$$\|u\|_{W^{2,2}_{cf}(U)} \leq C \left[ \|r^{2-\delta} \Delta(r^\delta u)\|_{W^{2,2}_{cf}(U')} + \|u\|_{L^2(U')} \right] \tag{10}$$

turns into

$$\|u\|_{W^{2,2}_\delta(U)} \leq C \left[ \|\Delta u\|_{W^{2,2}_{\delta-2}(U')} + \|u\|_{L^2_\delta(U')} \right].$$

## 4.1 Bounded geometry

In order to extend equation 10 to an estimate on the asymptotic space, we need to understand on which parameters $C$ depends. For example, for $\mathbb{R}^n$ the constant $C$ only depends on the ellipticity of the operator and on the relative size of the domains. In the above example this is also true: The conformally rescaled metric $g_{cf} = (\mathrm{d}\log r)^2 + g_{S^{n-1}}$ is a cylindrical metric and we can move any fixed domain by translation. By patching multiple local elliptic estimates we can establish a global elliptic estimate[19] like

$$\|u\|_{W^{2,2}_{cf}(\mathbb{R}^n\setminus K)} \leq C\left[\|r^{2-\delta}\Delta(r^\delta u)\|_{W^{2,2}_{cf}(\mathbb{R}^n\setminus K)} + \|u\|_{L^2(\mathbb{R}^n\setminus K)}\right].$$

In general, we study the behaviour of the constant $C$ by comparing the metric $g_{cf}$ with the flat metric. Namely, when using local coordinates, the dependence of $C$ can be found in the standard references on elliptic PDE's (e.g. Evans (1998) or Gilbarg & Trudinger (2001)). We only need to show that the Hölder and Sobolev norms induced by these local coordinates are equivalent to the ones induced by $g_{cf}$.

**Definition 4.1.** *Let $(M, g)$ be a Riemannian manifold, let $U \subset M$, $k, p \in \mathbb{N}$ and $\alpha \in (0, 1)$. The Sobolev space $W^{k,p}_g(U)$ is the space of all compactly supported functions on $U$ completed under the norm*

$$\|u\|^p_{W^{k,p}_g(U)} := \sum_{j=0}^{k} \|\nabla^j u\|^p_{L^p(U)}.$$

*The space $C^{k,\alpha}_g(U)$ is the space of all $k$ times differentiable functions satisfying*

$$\|u\|_{C^{k,\alpha}_g(U)} := \sum_{j=0}^{k} \sup_{x\in U} \|\nabla^j u(x)\|_g + \sum_{\substack{x,y\in U \\ d(x,y)<\mathrm{InjRad}_x(g)}} \frac{\|\nabla^k u(x) - \nabla^k u(y)\|_g}{d(x,y)^\alpha} < \infty$$

*where $d(x, y)$ is the geodesic distance between $x$ and $y$ and $\nabla^k u(y)$ is compared to $\nabla^k u(x)$ using parallel transport.*

*Remark* 4.2. Notice that for the Euclidean metric, this definition coincides with the standard definitions for Sobolev and Hölder spaces.

**Proposition 4.3.** *Let $(M, g)$ be a Riemannian manifold and $\{x_i\}$ local coordinates such that the coordinate metric $\delta$ and $g$ induce equivalent norms. If the first $k$ covariant derivatives of the connection form are bounded, then the $W^{k,p}$ and $C^{k,\alpha}$ norms that are induced by $g$ are equivalent to the ones induced by $\delta$.*

[19]This estimate can also be retrieved from the works of Lockhart & McOwen (1985) and of Melrose & Mendoza (1983)

*Proof.* Write $\nabla = \mathrm{d} + \Gamma$ where $\Gamma$ is the connection form. Because $\Gamma$ and its derivatives are bounded, $\|\nabla^j u\|_g \leq C \sum_{l \leq j} \|\mathrm{d}^l u\|_\delta$ for all $j = 0, \ldots, k$. By the same argument, the converse is also true. This shows the Sobolev norms are equivalent.

In order to compare the Hölder seminorms, we need to consider the variation of a tensor field along geodesics. That is, let $P_x^y$ be the parallel transport along the geodesic between $y$ and $x$ and $V = V^I \partial_I$ be a tensor field. Because

$$\frac{\|V^I(x)\partial_I - P_x^y V^I(y)\partial_I\|_g}{d(x,y)^\alpha} \leq C \frac{\|V^I(x)\partial_I - V^I(y)\partial_I\|_\delta}{d(x,y)^\alpha} + \|V^I(y)\| \cdot \frac{\|\partial_I - P_x^y \partial_I\|_g}{d(x,y)^\alpha},$$

the Hölder norms are equivalent if $\frac{\|\partial_I - P_x^y \partial_I\|_g}{d(x,y)^\alpha}$ is uniformly bounded. To show this, let $\gamma\colon [0, \mathrm{d}(x,y)] \to U$ be the geodesic between $x$ and $y$ parametrized by arclength and denote $P_{\gamma(0)}^{\gamma(t)} \partial_I$ in the local coordinates as $(P_{\gamma(0)}^{\gamma(t)} \partial_I)^J \partial_J$. By the parallel transport equation and the fundamental theorem of calculus,

$$\begin{aligned}
\|\partial_I - P_x^y \partial_I\|_g &= \left\| \int_{\mathrm{d}(x,y)}^0 \frac{\partial}{\partial s} \left( P_{\gamma(0)}^{\gamma(s)} \partial_I \right)^J \partial_J \,\mathrm{d}\, s \right\|_g \\
&= \left\| \int_0^{d(x,y)} \nabla_{\dot\gamma(s)} \partial_I \,\mathrm{d}\, s \right\|_g \\
&= \left\| \int_0^{d(x,y)} \left\| \nabla_{\dot\gamma(s)} \partial_I \right\| \mathrm{d}\, s \right\|_g
\end{aligned}$$

The connection form is given by the covariant derivative of $\partial_J$ and therefore, $\|\partial_I - P_x^y \partial_I\|_g \leq C \|\Gamma\|_{C_g^0(U)} \cdot d(x,y)$ for some constant $C > 0$. $\square$

Our first guess is to apply Proposition 4.3 using Riemann normal coordinates, because in these coordinates we have the Taylor expansion

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l + \mathcal{O}(|x|^3).$$

However, as explained by DeTurck & Kazdan (1981), normal coordinates are not the best coordinates for this purpose, but harmonic coordinates are. Harmonic coordinates are defined by the property $\Delta x_i = 0$ and are useful because in harmonic coordinates the Ricci curvature can be viewed as an elliptic operator acting on the metric, i.e.

$$\mathrm{Ric}_{ij} = -\frac{1}{2} g^{kl} \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l} + \dots .$$

In the following theorem we show the existence of harmonic coordinates and give a size estimate of the coordinate patches.

**Theorem 4.4** (Theorem 1.2 in Hebey (1999))**.** *Let $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $Q > 1$. Let $(M, g)$ be a Riemannian manifold whose injectivity radius is bounded below by some constant $i > 0$ and suppose there exists a constant $C > 0$ such that*

$$\|\nabla^j \mathrm{Ric}\,\|_{C^0_g(M)} \leq C$$

*for all $j = 0, \dots, k$. There exists a constant $r_H > 0$ such that for any $p \in M$, there are harmonic coordinates $\{x_i\}$ on $B_r(p)$ that satisfy $Q^{-1}\delta_{\mu\nu} < g_{\mu\nu} < Q\,\delta_{\mu\nu}$ as bilinear forms, and*

$$\sum_{1<j<k} r_H^j \sup_{y \in B_r(p)} |\partial^{(j)} g(y)| + r_H^{k+\alpha} \sup_{\substack{y,z \in B_r(p) \\ y \neq z}} \frac{|\partial^{(k)} g(y) - \partial^{(k)} g(z)|}{|y-z|^\alpha} < Q - 1.$$

*Remark* 4.5. Recall that the injectivity radius estimates the largest ball on which the Riemann normal coordinates are defined. Similarly, the constant $r_H$ in Theorem 4.4 estimates the largest ball on which the harmonic coordinates are defined and have $C^{k,\alpha}$ control on the metric. Therefore, the constant $r_H$ is referred as the harmonic radius in literature.

Theorem 4.4 is the key theorem we use to find elliptic estimates on our Riemannian manifold. We already have seen we can get elliptic estimates on our Riemannian manifold by considering elliptic estimates inside a coordinate chart and converting the Euclidean norms to the Riemannian norms. Theorem 4.4 assures us the existence of sufficiently large coordinate charts which fulfils all the prerequisites of Proposition 4.3. Moreover, it gives us explicit bounds of the metric in terms of the injectivity radius and (the derivatives of) the Ricci curvature. This implies that the constants in the elliptic estimates will also be bounded by these quantities. Therefore, if we can bound the injectivity radius and (the derivatives of) the Ricci curvature uniformly in $x \in M$ and in the collapsing parameter $\epsilon$, our estimates will be uniform in $x$ and $\epsilon$.

To set up the analysis on the asymptotic region of $P$, we define the following notation:

**Definition 4.6.** *Let $\pi\colon P \to B'$ be the principal circle bundle on the Gibbons-Hawking space and fix a large $R_0 > 0$. We define the asymptotic part of $B'$ as $[R_0, \infty) \times \Sigma \subset B'$, where $\Sigma$ is $S^2$ or $T^2$ when $B = \mathbb{R}^3$ or $B \neq \mathbb{R}^3$ respectively. We denote the asymptotic part of $P$ as*

$$P_\infty := P_\infty(R_0) := \pi^{-1}([R_0, \infty) \times \Sigma).$$

**Definition 4.7.** *Let $g^{GH} = h_\epsilon g_B + \epsilon^2 h_\epsilon^{-1} \eta^2$ be the Gibbons-Hawking metric. Define*

$$\Omega = \begin{cases} h_\epsilon^{-\frac{1}{2}} & \text{if } B = \mathbb{R} \times T^2 \\ r^{-1} h_\epsilon^{-\frac{1}{2}} & \text{otherwise,} \end{cases}$$

*where $r$ is the Euclidean distance on $\mathbb{R}^3$, $\mathbb{R}^2$ or $\mathbb{R}$ when $B = \mathbb{R}^3$, $B = \mathbb{R}^2 \times S^1$ or $B = \mathbb{R} \times T^2$ respectively. We define the conformally rescaled metric $g_{cf}$ as*

$$g_{cf} = \Omega^2 \cdot g^{GH}.$$

The difference between the case $B = \mathbb{R} \times T^2$ and the rest is due to the fact that the harmonic functions on $\mathbb{R} \times T^2$ have exponential rather than polynomial growth or decay. The conformal rescaling of $h_\epsilon^{-1}$ in $\Omega$ is convenient, because for $S^1$ invariant functions the analysis reduces to the standard analysis on $\mathbb{R}^n$. This is due to the fact that $h_\epsilon \Delta^{GH} = \Delta^B$ for $S^1$ invariant functions. In order to apply Theorem 4.4, we need to calculate the Ricci tensor for our metric. The metric $g^{GH}$ is hyperkähler and hence it is Ricci flat. However, we consider the conformally rescaled metric $g_{cf}$, which is not hyperkähler.

**Lemma 4.8.** *Consider the Gibbons-Hawking manifold $P$ with the metric $g_{cf}$. The Ricci curvature tensor and its first $k$ covariant derivatives are given in terms of $\mathrm{d}\log\Omega$ and its $k+1$ covariant derivatives. In particular, on $P_\infty$ these are uniformly bounded for $\epsilon \in (0,1)$.*

*Proof.* According to Theorem 1.159 in Besse (1987), the Ricci curvature of $g_{cf}$ is given by

$$\operatorname{Ric}(g_{cf}) = \operatorname{Ric}(g^{GH}) - 2(\operatorname{Hess}(\log\Omega) + \mathrm{d}\log\Omega \otimes \mathrm{d}\log\Omega) - (\Delta_{cf}\log\Omega - 2|\mathrm{d}\log\Omega|^2_{cf})g,$$

where the Hessian is calculated with respect to $g_{cf}$. Because the Gibbons-Hawking metric is Ricci-flat, $\nabla^k \operatorname{Ric}(g_{cf})$ is bounded by $\mathrm{d}\log\Omega$ and its first $k+1$ derivatives.

In order to estimate (the derivatives of) $\mathrm{d}\log\Omega$ at infinity, we need to calculate the derivatives of $\mathrm{d}\log h_\epsilon$ (and $\mathrm{d}\log r$ when $B \neq \mathbb{R}\times T^2$). To calculate $\nabla^j \mathrm{d}\log r$, we first calculate the covariant derivative using the co-frames

$$\begin{array}{lllll} \{\mathrm{d}\log r, & \mathrm{d}\,\theta, & \sin\theta\,\mathrm{d}\,\phi, & \frac{\epsilon}{rh_\epsilon}\eta\} & \text{if} \quad B = \mathbb{R}^3, \\ \{\mathrm{d}\log r, & \mathrm{d}\,\theta, & r^{-1}\,\mathrm{d}\,\phi, & \frac{\epsilon}{rh_\epsilon}\eta\} & \text{if} \quad B = \mathbb{R}^2\times S^1, \\ \{\mathrm{d}\,r, & \mathrm{d}\,\theta, & \mathrm{d}\,\phi, & \frac{\epsilon}{h_\epsilon}\eta\} & \text{if} \quad B = \mathbb{R}\times T^2. \end{array}$$

For uniformity, we denote $\rho = \log r$ when $B \neq \mathbb{R}\times T^2$ and $\rho = r$ otherwise. According to the Koszul formula, the connection form is given in term of the the exterior derivatives and the non-zero terms are

$$\begin{aligned} \mathrm{d}(\sin\theta\,\mathrm{d}\,\phi) &= \frac{1}{\tan(\theta)}\,\mathrm{d}\,\theta\wedge\sin(\theta)\,\mathrm{d}\,\phi && \text{if } B = \mathbb{R}^3 \\ \mathrm{d}(e^{-\rho}\,\mathrm{d}\,\phi) &= -\,\mathrm{d}\,\rho\wedge e^{-\rho}\,\mathrm{d}\,\phi && \text{if } B = \mathbb{R}^2\times S^1 \\ \mathrm{d}(\epsilon e^{-\rho}h_\epsilon^{-1}\eta) &= -\,(\mathrm{d}\,\rho + \mathrm{d}\log h_\epsilon)\wedge(\epsilon e^{-\rho}h_\epsilon^{-1}\eta) \\ &\quad + *^{cf}(\mathrm{d}\log h_\epsilon\wedge\epsilon e^{-\rho}h_\epsilon^{-1}\eta) && \text{if } B \neq \mathbb{R}\times T^2 \\ \mathrm{d}(\epsilon h_\epsilon^{-1}\eta) &= -\,\mathrm{d}\log h_\epsilon\wedge(\epsilon h_\epsilon^{-1}\eta) + *^{cf}(\mathrm{d}\log h_\epsilon\wedge\epsilon h_\epsilon^{-1}\eta) && \text{if } B = \mathbb{R}\times T^2. \end{aligned}$$

This implies that there are some constants $C_1, C_2 > 0$ such that

$$\|\nabla^j\,\mathrm{d}\,\rho\|_{cf} \leq C_1 + C_2\sum_{i=0}^{j-1}\|\nabla^i\,\mathrm{d}\log h_\epsilon\|_{cf}.$$

In order to estimate $\mathrm{d}\log h_\epsilon = \frac{1}{h_\epsilon}\,\mathrm{d}\,h_\epsilon$, recall from Lemma 3.4 that $h_\epsilon^{-1}$ is bounded below, and hence, by repeated use of the Leibniz rule, one can show

$$\|\nabla^j\,\mathrm{d}\log h_\epsilon\|_{cf} \leq C_3\sum_{l=0}^{j}\|\nabla^l\,\mathrm{d}\,h_\epsilon\|_{cf}$$

for some constant $C_3 > 0$. Moreover, Lemma 3.4 shows that $h_\epsilon$ is exponentially close to

$$\begin{cases} 1+\epsilon\beta e^{-\rho} & \text{if } B=\mathbb{R}^3 \\ 1+\epsilon\beta\rho & \text{otherwise} \end{cases}$$

for some $\beta > 0$. Hence $\nabla^k \,\mathrm{d}\log h_\epsilon$ is bounded by the first k derivatives of $\mathrm{d}\,\rho$, which are bounded by the first $k-1$ derivatives of $\mathrm{d}\log h_\epsilon$. By induction, $\nabla^k \,\mathrm{d}\log h_\epsilon$ must be uniformly bounded. □

Before we continue our study in the bounded geometry of $P_\infty$, we first consider the weighted operator and study its analytic properties:

**Definition 4.9.** *Let $\Omega$ be as described in Definition 4.7. Consider*

$$\rho = \begin{cases} r & \textit{if } B=\mathbb{R}\times T^2 \\ \log r & \textit{otherwise,} \end{cases}$$

*where $r$ is the Euclidean distance on $\mathbb{R}^3$, $\mathbb{R}^2$ or $\mathbb{R}$ when $B=\mathbb{R}^3$, $B=\mathbb{R}^2\times S^1$ or $B=\mathbb{R}\times T^2$ respectively. For all $\delta\in\mathbb{R}$, We define the weighted operator $L_\delta$ as*

$$L_\delta = e^{-\delta\rho}\Omega^{-2}\Delta^{GH}(e^{\delta\rho}\ldots).$$

As shown in the proof of Lemma 4.8, $\mathrm{d}\,\rho$ has norm one and all its derivatives are bounded uniformly for $\epsilon\in(0,1)$. Therefore, we use this as the radial parameter by which we will decay. We use the bounds on $\mathrm{d}\,\rho$ and its higher derivatives to show that $L_\delta$ is strictly elliptic in the sense of Gilbarg & Trudinger (2001):

**Proposition 4.10.** *For each $\delta\in\mathbb{R}$, the operator $L_\delta$ is a strictly elliptic operator with bounded coefficients between $C^{k+2,\alpha}_{cf}$ and $C^{k,\alpha}_{cf}$, uniformly in $\epsilon\in(0,1)$. That is, if one considers the local coordinates given in Theorem 4.4 and expands $L_\delta$ as*

$$L_\delta = a^{ij}\partial_i\partial_j + b^i\partial_i + c,$$

*then there exist $\lambda,\Lambda>0$, independent of $\epsilon$, such that*

$$-a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall\xi\in\mathbb{R}^4$$

*and $\|a^{ij}\|_{C^{0,\alpha}_{cf}}, \|b^i\|_{C^{0,\alpha}_{cf}}, \|c\|_{C^{0,\alpha}_{cf}} < \Lambda$.*

*Proof.* For any twice differentiable function $u$, one can show

$$\begin{aligned} e^{-\delta\rho}\Omega^{-2}\Delta^{GH}(e^{\delta\rho}u) =& \Delta_{cf}u + 2\langle \mathrm{d}\log\Omega - \delta\,\mathrm{d}\,\rho, \mathrm{d}\,u\rangle_{cf} \\ &+ u\cdot\left(-\delta^2 - \delta\,\nabla^*_{cf}\,\mathrm{d}\,\rho + 2\delta\langle \mathrm{d}\log\Omega, \mathrm{d}\,\rho\rangle_{cf}\right)\cdot u. \end{aligned}$$

Therefore, $L_\delta$ differs from $\Delta_{cf}$ by a first order differential operator. The Laplacian is always strictly elliptic and so we have left to show that $L_\delta - \Delta_{cf}$ has bounded coefficients. This is true if (the derivatives of) $\mathrm{d}\log\Omega$ and $\mathrm{d}\,\rho$ are bounded. In the proof of Lemma 4.8 this is shown explicitly. □

We return to the study of the bounded geometry of $P_\infty$. Except for the injectivity radius, all conditions stated in Theorem 4.4 are satisfied by Lemma 4.8. However, in most cases the injectivity radius decays to zero. This is because the circle radius of the fibers is $\frac{2\pi\epsilon}{rh_\epsilon}$ or $\frac{2\pi\epsilon}{h_\epsilon}$ respectively. To remedy this, we replace the fibers with their universal cover. To be precise, we will consider local trivialisations over sufficiently large, contractible open sets and we work on the universal cover over these trivialisations. We will show that, on these local universal covers, the injectivity radius is bounded below. For this we use a result by Cheeger et al. (1982), which states that it is sufficient to get a lower bound on $\mathrm{Vol}_{cf}(B_1(p))$ for all $p \in P_\infty$. Secondly, we will determine how the Sobolev and Hölder norms change when we project them back to neighbourhoods on $P_\infty$.

**Lemma 4.11.** *On local universal covers of $P_\infty$, the injectivity radius is bounded below, uniformly in $\epsilon \in (0,1)$.*

*Proof.* We explain the case $B = \mathbb{R}\times T^2$. Pick $p = (x_0, 0) \in P_\infty$ and choose $\varrho > 0$ such that the ball $B_\varrho(x_0) \subset B$ is contractible. Next, we trivialise $P|B_\varrho(x_0) \simeq B_\varrho(x_0)\times S^1$ and consider the following rectangular neighbourhood on its universal cover:

$$R_\varrho(p) := \left\{(x,t) \in B_\varrho(x_0)\times\mathbb{R} \colon |t| < \frac{h_\epsilon(x)}{\epsilon}\varrho\right\}$$

We claim $R_\varrho(p)$ lies inside a circumscribed ball of fixed length. To show this, pick $(x,t) \in R_\rho(p)$ and consider the path that goes parallel along the coordinate axis. According to the gauge fix in Lemma 3.12, the length of this path is bounded above by some uniform constant $C > 0$, and so $R_\varrho(p)$ lies inside the ball of radius $C$ centred at $p$. The volume of $R_\varrho(p)$ is equal to

$$\mathrm{Vol}_{cf}(R_\varrho(p)) = \int_{x\in B_\varrho(x_0)} \int_{-\frac{h_\epsilon}{\epsilon}\varrho}^{\frac{h_\epsilon}{\epsilon}\varrho} \frac{\epsilon}{h_\epsilon} \mathrm{Vol}_{g_B} \wedge \mathrm{d}\, t = 2\,\mathrm{Vol}_{g_B}(B_\rho(x_0)) = \frac{8}{3}\pi\varrho^3.$$

According to Cheeger et al. (1982), the injectivity radius at $p$ on $R_\rho(p)$ is bounded below uniformly in $\epsilon \in (0,1)$.

When $B = \mathbb{R}^2 \times S^1$ the injectivity radius will still decay to zero at infinity. This is due to the term $\frac{1}{e^\rho} g_{S^1}$ in the metric. However, when we consider $P_\infty$ as a $T^2$ bundle and use this unwrapping trick for both decaying fibers at the same time, we still get a lower bound on the injectivity radius. □

Any function on $P_\infty$ can be lifted to periodic functions on these local universal covers. This gives us two ways to measure these functions: We can measure them using the Sobolev or Hölder norms on $P_\infty$ or using the respective norms on the local universal cover. Due to the $S^1$ invariance of the metric, we claim that their Hölder norms are equivalent.

**Lemma 4.12.** *Let $V \subseteq P_\infty$ be open such that $V$ restricts to a trivial $S^1$-bundle or $T^2$-bundle respectively over a contractible base space. Let $\hat{V}$ be the universal cover of $V$. Then, for any $u \in C^{k,\alpha}_{cf}(V)$,*

$$\|u\|_{C^{k,\alpha}_{cf}(V)} = \|\hat{u}\|_{C^{k,\alpha}_{cf}(\hat{V})},$$

*where $\hat{u}$ is the lift of $u$ in $C^{k,\alpha}_{cf}(\hat{V})$.*

In the Sobolev case, the norms are not equivalent. Namely, a ball on a local universal cover will contain multiple copies of the fundamental domain. Because the circle radius may shrink to zero at infinity, the number of fundamental domains may not be bounded. However, we can solve this issue by changing the volume form.

**Lemma 4.13.** *Let $r > 0$ be less than the injectivity radius found in Lemma 4.11. Let $p \in P_\infty$, let $B_r(p)$ be the ball of radius $r$ in $P_\infty$ and let $\hat{B}_r(p)$ be the ball of* *radius $r$ on the local universal cover of $P_\infty$ at $p$. Consider the function*

$$v^2 = \begin{cases} \frac{e^\rho h_\epsilon}{\epsilon} & \text{if } B = \mathbb{R}^3 \\ \frac{e^{2\rho} h_\epsilon}{\epsilon} & \text{if } B = \mathbb{R}^2 \times S^1 \\ \frac{h_\epsilon}{\epsilon} & \text{otherwise.} \end{cases}$$

*Then, there exist $1 < M_1 < M_2$ and $0 < C_1 < C_2$, independent of $p$ and $\epsilon$, such that for all $u \in L^2(B_r(p))$*

$$C_1 \|v \cdot u\|_{L^2(B_{r/M_2}(p))} \leq \|\hat{u}\|_{L^2(\hat{B}_{r/M_1}(p))} \leq C_2 \|v \cdot u\|_{L^2(B_r(p))},$$

*where $\hat{u}$ is the periodic lift of $u$ on $\hat{B}_r(x)$.*

*Proof.* We explain the case $B = \mathbb{R} \times T^2$. For the volume estimates, we again work with the rectangular domains, described in the proof of Lemma 4.11,

$$R_\varrho(p) := \left\{ (x,t) \in B_\varrho(x_0) \times \mathbb{R} \colon |t| < \frac{h_\epsilon(x)}{\epsilon} \varrho \right\},$$

where $p = (x_0, 0)$ in a local trivialisation. As explained in the proof of Lemma 4.11, each rectangular domain lies inside a circumscribed ball whose radius only depends on $\varrho$. We claim it also has an inscribed ball with the same property. Indeed, to estimate the radius of the inscribed ball one picks an $(x_1, t_1) \in \hat{B}_{\tilde{\varrho}}(x_0, t_0)$ for some $\tilde{\varrho} > 0$ and a geodesic $\gamma$ between $(x_0, t_0)$ and $(x_1, t_1)$. To show that $(x_1, t_1)$ lies inside $R_\varrho(x_0, 0)$ one needs to calculate the coordinates of $x_1$. For example, to calculate the $\rho$ coordinate of $x_1$ one uses

$$\rho(x_1) = \int_0^1 \mathrm{d}\,\rho(\dot{\gamma}(t))\,\mathrm{d}\,t \leq \int_0^1 \|\mathrm{d}\,\rho\| \cdot \|\dot{\gamma}(t)\|\,\mathrm{d}\,t = \mathrm{Length}(\gamma) < \tilde{\varrho}.$$

The radii of the inscribed and circumscribed balls determine the values of $M_1$ and $M_2$. Using these balls, we only need to show

$$C_1 \|v \cdot u\|^2_{L^2(F_\varrho(p))} \leq \|\hat{u}\|^2_{L^2(R_\varrho(p))} \leq C_2 \|v \cdot u\|^2_{L^2(F_\varrho(p))},$$

where $F_\varrho(p)$ is the fundamental domain

$$F_\varrho(p) := B_\varrho(x_0) \times [-\pi, \pi].$$

Let us estimate $\|\hat{u}\|^2_{L^2(R_\varrho(p))}$. Because the circle radius of the fiber is $\frac{2\pi\epsilon}{h_\epsilon}$, a rectangular domain contains a number of fundamental domains between $\lfloor \min_{R_\varrho} \frac{h_\epsilon}{2\pi\epsilon}\varrho \rfloor$ and $\lceil \max_{R_\varrho} \frac{h_\epsilon}{2\pi h_\epsilon}\varrho \rceil$. Therefore, by the periodicity of $\hat{u}$,

$$\lfloor \min_{R_\varrho(p)} \frac{h_\epsilon}{2\pi\epsilon}\varrho \rfloor \cdot \|u\|^2_{L^2(F_\varrho(p))} \leq \|\hat{u}\|^2_{L^2(R_\varrho(p))} \leq \lceil \max_{R_\varrho(p)} \frac{h_\epsilon}{2\pi\epsilon}\varrho \rceil \cdot \|u\|^2_{L^2(F_\varrho(p))}.$$

We can find $C_1$ and $C_2$ by bounding $\lfloor \min_{R_\varrho} \frac{h_\epsilon}{2\pi\epsilon}\varrho \rfloor$ and $\lceil \max_{R_\varrho} \frac{h_\epsilon}{2\pi\epsilon}\varrho \rceil$ on these domains. Asymptotically $h_\epsilon$ is approximated by $1 + \epsilon(\beta\rho + \mathcal{O}(e^{-\rho}))$ for some $\beta > 0$, and hence

$$\frac{\lfloor \min_{R_\varrho(p)} \frac{h_\epsilon}{2\pi\epsilon}\varrho \rfloor}{\frac{h_\epsilon}{2\pi\epsilon}} \leq 1 + \frac{\min h_\epsilon - h_\epsilon}{h_\epsilon} = 1 - \epsilon\frac{\beta\varrho + \mathcal{O}(e^{-\rho})}{1 + \epsilon(\beta\rho + \mathcal{O}(e^{-\rho}))} \leq 1 - \epsilon\beta\varrho + \mathcal{O}(e^{-\rho}).$$

A lower bound and the bounds for $\lceil \max_{R_\varrho(p)} \frac{h_\epsilon}{2\pi\epsilon}\varrho \rceil$ can be found similarly. Therefore, there exist some constants $C_1, C_2 > 0$ such that

$$C_1\|v \cdot u\|^2_{L^2(F_\varrho(p))} \leq \|\hat{u}\|^2_{L^2(R_\varrho(p))} \leq C_2\|v \cdot u\|^2_{L^2(F_\varrho(p))}.$$

□

As discussed before, the study of weighted operators is equivalent to the study of weighted norms. Inspired by the form of $L_\delta$ and Lemma 4.13, we define the following weighted norm.

**Definition 4.14.** *Let $\Omega$ and $g_{cf}$ be as described in Definition 4.7 and let $\rho$ be as in Definition 4.9. For any $k \in \mathbb{N}$, $\alpha \in (0,1)$, $\delta \in \mathbb{R}$, we define the weighted Hölder norm on $U \subseteq P_\infty$ as*

$$\|u\|_{C^{k,\alpha}_\delta(U)} = \|e^{-\delta\rho} \cdot u\|_{C^{k,\alpha}_{cf}(U)}.$$

*For any $k \in \mathbb{N}$, $\delta \in \mathbb{R}$, we define the weighted $L^2$ and Sobolev norm on $U \subseteq P_\infty$ as*

$$\begin{aligned} \langle u, v\rangle_{L^2_\delta(U)} &= \langle e^{-\delta\rho}\, u, e^{-\delta\rho}\, v\rangle_{\tilde{L}^2(U)} \\ \|u\|^2_{W^{k,2}_\delta(U)} &= \sum_{n=0}^{k} \|\, |\nabla^n(e^{-\delta\rho} \cdot u)|_{cf}\|^2_{\tilde{L}^2(U)} \end{aligned}$$

*where $\tilde{L}^2(U)$ is the $L^2$ norm with respect to the volume form*

$$\widetilde{\mathrm{Vol}} := \begin{cases} \mathrm{d}\,\rho \wedge \mathrm{Vol}_{S^2} \wedge \eta & \text{if } B = \mathbb{R}^3 \\ \mathrm{d}\,\rho \wedge \mathrm{Vol}_{S^1 \times S^1} \wedge \eta & \text{if } B = \mathbb{R}^2 \times S^1 \\ \mathrm{d}\,\rho \wedge \mathrm{Vol}_{T^2} \wedge \eta & \text{if } B = \mathbb{R} \times T^2. \end{cases}$$

## 4.2 Weighted local elliptic estimates

With all these ingredients we now have a method to establish elliptic estimates on $P_\infty$. For example, to rephrase the estimate[20]

$$\|u\|_{C^{k,\alpha}_{\mathbb{R}^n}} \leq C \left[ \|\Delta u\|_{C^{k-2,\alpha}_{\mathbb{R}^n}} + \|u\|_{C^0} \right],$$

we follow the following steps:

*Step 1:* First we pick the radii of the balls on which we apply the estimates. Hence, for any $x \in P_\infty$, we consider a neighbourhood $V \subseteq P_\infty$ which restricts to a trivial $S^1$- (resp. $T^2$-) bundle over a contractible base space. According to Lemma 4.11 we can pick $V$ large enough such that the injectivity radius is bounded below uniformly for $x \in P_\infty$ and $\epsilon \in (0,1)$. According to Theorem 4.4 there exists $r_H > 0$ uniformly, such that $B_{r_H}(x)$ can be equipped with harmonic coordinates. We pick $0 < r < r' < r_H$.

*Step 2:* Fix $k \in \mathbb{N}_{\geq 2}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$. Pick $u \in C^{k,\alpha}_\delta(B_{r'}(x))$ and consider $\|u\|_{C^{k,\alpha}_\delta(B_r(x))}$. By Definition 4.14 we write

$$\|u\|_{C^{k,\alpha}_\delta(B_r(x))} = \|e^{-\delta\rho} u\|_{C^{k,\alpha}_{cf}(B_r(x))}.$$

We picked our radii such that $B_r(x)$ can be equipped with harmonic coordinates. Moreover, by Proposition 4.3 we can view the Hölder norm on the right hand side as the Hölder norm induced by this chart.

*Step 3:* Next, we lift $u \in B_{r'}(x)$ to a periodic function $\hat{u}$ on the local universal cover

[20] This regularity result is a combination of Gilbarg & Trudinger (2001), Problem 6.1, Theorem 9.19 and Folland (1995) Theorem 6.33.

inside $\hat{B}_{r'}(x)$. By Lemma 4.12,

$$\|e^{-\delta\rho}u\|_{C^{k,\alpha}_{cf}(B_r(x))} = \|e^{-\delta\rho}\hat{u}\|_{C^{k,\alpha}_{cf}(\hat{B}_r(x))}.$$

Because the local universal cover has bounded geometry, we are able to apply the elliptic estimate

$$\|e^{-\delta\rho}\hat{u}\|_{C^{k,\alpha}_{cf}(\hat{B}_r(x))} \leq C\left[\|L_\delta(e^{-\delta\rho}\hat{u})\|_{C^{k-2,\alpha}_{cf}(\hat{B}'_r(x))} + \|e^{-\delta\rho}\hat{u}\|_{C^0(\hat{B}'_r(x))}\right].$$

*Step 4:* Using the fact that $L_\delta$ is invariant under deck transformations, we project down to balls on $P_\infty$, and by Lemma 4.12:

$$\|e^{-\delta\rho}u\|_{C^{k,\alpha}_{cf}(B_r(x))} \leq C\left[\|L_\delta(e^{-\delta\rho}u)\|_{C^{k-2,\alpha}_{cf}(B'_r(x))} + \|e^{-\delta\rho}u\|_{C^0(B'_r(x))}\right].$$

Using the definition of $L_\delta$,

$$\|e^{-\delta\rho}u\|_{C^{k,\alpha}_{cf}(B_r(x))} \leq C\left[\|e^{-\delta\rho}\Omega^{-2}\Delta^{GH}u\|_{C^{k-2,\alpha}_{cf}(B'_r(x))} + \|e^{-\delta\rho}\hat{u}\|_{C^0(B'_r(x))}\right],$$

and we conclude:

**Theorem 4.15.** *Let $k \in \mathbb{N}_{\geq 2}$, $\delta \in \mathbb{R}$ and $\alpha \in (0,1)$. For sufficiently small $0 < r < r'$, there exists an uniform constant $C > 0$ such that for all $x \in P_\infty$ and any distribution $u$ on $B_{r'}(x)$ with $\Omega^{-2}\Delta^{GH}u \in C^{k-2,\alpha}_\delta(B_{r'}(x))$,*

$$u \in C^{k,\alpha}_\delta(B_{r'}(x))$$

*and*

$$\|u\|_{C^{k,\alpha}_\delta(B_r(x))} \leq C\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{k-2,\alpha}_\delta(B_{r'}(x))} + \|u\|_{C^0_\delta(B_{r'}(x))}\right].$$

Similarly, we get a local Schauder estimate using Sobolev norms. For this we use the results on $\mathbb{R}^n$ from Evans (1998) (Theorem 1 in section 6.3.1) and Bandle & Flucher (1998) (Theorem 7-12).

**Theorem 4.16.** *Let $k \in \mathbb{N}_{\geq 2}$ and $\delta \in \mathbb{R}$. For sufficiently small $0 < r < r'$, there exists an uniform constant $C > 0$ such that for all $x \in P_\infty$ and any distribution $u$ on $B_{r'}(x)$ with $\Omega^{-2}\Delta^{GH}u \in C^{k-2,\alpha}_\delta(B_{r'}(x))$,*

$$u \in W^{k,2}_\delta(B_{r'}(x))$$

*and*

$$\|u\|^2_{W^{k,2}_\delta(B_r(x))} \leq C\left[\|\Omega^{-2}\Delta^{GH}u\|^2_{W^{k-2,2}_\delta(B_{r'}(x))} + \|u\|^2_{L^2_\delta(B_{r'}(x))}\right].$$

**Theorem 4.17.** *Let $\delta \in \mathbb{R}$ and $\alpha \in (0,1)$. For sufficiently small $0 < r < r'$, there exists an uniform constant $C > 0$ such that for all $x \in P_\infty$ and any $u \in C^{2,\alpha}_\delta(B_{r'}(x))$,*

$$\|u\|_{C^{0,\alpha}_\delta(B_r(x))} \leq C\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{0,\alpha}_\delta(B_{r'}(x))} + \|u\|_{L^2_\delta(B_{r'}(x))}\right].$$

## 4.3 Weighted asymptotic elliptic estimates

As shown in sections 8 to 10 of Pacard (2008), Fredholmness of an elliptic operator can be shown using global Schauder estimates. In this section we prove similar estimates on $P_\infty$. In order to describe these global estimates we need to slightly increase the size of $P_\infty$. For this, recall from Definition 4.6 that $P_\infty := \pi^{-1}([R_0,\infty)\times\Sigma)$ for some compact manifold $\Sigma$ and large $R_0 > 0$. Pick $R_1$ slightly smaller than $R_0$ and define the slightly larger $P'_\infty$ as

$$P'_\infty := \pi^{-1}([R_1,\infty)\times\Sigma).$$

Their difference is given by

$$K_\infty := \pi^{-1}([R_1,R_0]\times\Sigma).$$

**Theorem 4.18.** *Let $k \in \mathbb{N}_{\geq 2}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$. There exists a uniform constant $C > 0$ such that for any bounded $u \in C^{k,\alpha}_{loc}(P'_\infty/\mathbb{Z}_2)$ with $\Omega^{-2}\Delta^{GH}u \in C^{k-2,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)$,*

$$u \in C^{k,\alpha}_\delta(P_\infty/\mathbb{Z}_2)$$

*and*

$$\|u\|_{C^{k,\alpha}_\delta(P_\infty/\mathbb{Z}_2)} \leq C\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{k-2,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{C^0_\delta(P'_\infty/\mathbb{Z}_2)}\right].$$

*Furthermore, if $u$ vanishes on $\partial P'_\infty/\mathbb{Z}_2$, then*

$$\|u\|_{C^{k,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} \leq C\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{k-2,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{C^0_\delta(P'_\infty/\mathbb{Z}_2)}\right].$$

*Proof.* Let $r$ and $r'$ be as described in the steps for Theorem 4.15. Because $u \in C^{k,\alpha}_{loc}(P'_\infty)$, $u$ must lie in $C^{k,\alpha}_\delta(B_{r'}(x))$ for all $x \in P_\infty$. Theorem 4.15 states that

$$\begin{aligned}\|u\|_{C^{k,\alpha}_\delta(B_r(x))} \leq & C\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{k-2,\alpha}_\delta(B_{r'}(x))} + \|u\|_{C^0_\delta(B_{r'}(x))}\right] \\ \leq & C\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{k-2,\alpha}_\delta(P'_\infty)} + \|u\|_{C^0_\delta(P'_\infty)}\right].\end{aligned}$$

Varying $x \in P_\infty$, we conclude that

$$\|u\|_{C^{k,\alpha}_\delta(P_\infty)} = \sup_{x\in P_\infty} \|u\|_{C^{k,\alpha}_\delta(B_r(x))} \leq C\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{k-2,\alpha}_\delta(P'_\infty)} + \|u\|_{C^0_\delta(P'_\infty)}\right].$$

By working $\mathbb{Z}_2$ equivariantly on $P_\infty$, the first estimate follows.

For the boundary regularity estimate we use the same method, combined with Corollary 6.7 from Gilbarg & Trudinger (2001), which states that, for any $x$ close to the boundary,

$$\|u\|_{C^{k,\alpha}_\delta(B_r(x)\cap P'_\infty)} \leq C\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{k-2,\alpha}_\delta(P'_\infty)} + \|u\|_{C^0_\delta(P'_\infty)}\right].$$

$\square$

Using the same method one can extend Theorem 4.17 to a global version. For Theorem 4.16, we need to use a summation method, similarly to Proposition 6.1.1 in Pacard (2008). Namely, we pick $\kappa > 0$ and write $P'_\infty$ as the union of annuli $A_n := \pi^{-1}(B_{R_0+\kappa(n+1)} \setminus B_{R_0+\kappa n})$, and we sum the estimates for all annuli. Because the radius of the circle fiber is uniformly bounded above we can cover $A_n$ with a fixed number of balls such that on each ball we can apply Theorem 4.16. For large enough $\kappa$, we get the estimate

$$\|u\|^2_{W^{k,2}_\delta(A_n)} \leq C \sum_{m=n-1}^{n+1} \left[\|\Omega^{-2}\Delta^{GH}u\|^2_{W^{k-2,2}_\delta(A_m)} + \|u\|^2_{L^2_\delta(A_m)}\right]$$

for all $n \in \mathbb{Z}_{\geq 0}$. Taking the union over the first $N$ annuli yields

$$\begin{aligned}\|u\|_{W^{k,2}_\delta\left(\bigcup_{n=1}^N A_n\right)} \leq & 3C \sum_{n=0}^{N+1} \left[\|\Omega^{-2}\Delta^{GH}u\|_{W^{k-2,2}_\delta(A_n)} + \|u\|_{L^2_\delta(A_n)}\right] \\ \leq & 3C\left[\|\Omega^{-2}\Delta^{GH}u\|_{W^{k-2,2}_\delta\left(\bigcup_{n=0}^{N+1} A_n\right)} + \|u\|_{L^2_\delta\left(\bigcup_{n=0}^{N+1} A_n\right)}\right].\end{aligned}$$

If one assumes that $u$ vanishes on the boundary of $P'_\infty$,

$$\|u\|_{W^{k,2}_\delta(\bigcup_{n=0}^N A_n)} \leq 3C \left[ \|\Omega^{-2}\Delta^{GH} u\|_{W^{k-2,2}_\delta(\bigcup_{n=0}^{n+1} A_n)} + \|u\|_{L^2_\delta(\bigcup_{n=0}^{n+1} A_n)} \right].$$

Taking the limit $N \to \infty$ we conclude:

**Theorem 4.19.** *Let $k \in \mathbb{N}_{\geq 2}$ and $\delta \in \mathbb{R}$. There exists a uniform constant $C > 0$ such that for any $L^2_\delta$-bounded $u \in W^{k,2}_{loc}(P'_\infty/\mathbb{Z}_2)$ with $\Omega^{-2}\Delta^{GH} u \in W^{k-2,2}_\delta(P'_\infty/\mathbb{Z}_2)$,*

$$u \in W^{k,2}_\delta(P_\infty/\mathbb{Z}_2)$$

*and*

$$\|u\|_{W^{k,2}_\delta(P_\infty/\mathbb{Z}_2)} \leq C \left[ \|\Omega^{-2}\Delta^{GH} u\|_{W^{k-2,2}_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} \right].$$

*Furthermore, if $u$ vanishes on $\partial P'_\infty/\mathbb{Z}_2$, then*

$$\|u\|_{W^{k,2}_\delta(P'_\infty/\mathbb{Z}_2)} \leq C \left[ \|\Omega^{-2}\Delta^{GH} u\|_{W^{k-2,2}_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} \right].$$

**Theorem 4.20.** *Let $k \in \mathbb{N}_{\geq 2}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$. There exists a uniform constant $C > 0$ such that for any $L^2_\delta$-bounded $u \in L^2(P'_\infty/\mathbb{Z}_2)$ with $\Omega^{-2}\Delta^{GH} u \in C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)$,*

$$u \in C^{0,\alpha}_\delta(P_\infty/\mathbb{Z}_2)$$

*and*

$$\|u\|_{C^{0,\alpha}_\delta(P_\infty/\mathbb{Z}_2)} \leq C \left[ \|\Omega^{-2}\Delta^{GH} u\|_{C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} \right].$$

These estimates do not imply Fredholmness. However, as in Pacard (2008), if in Theorem 4.19 we can change $\|u\|_{L^2_\delta(P'/\mathbb{Z}_2)}$ into $\|u\|_{L^2_\delta(K)}$ for some compact set $K$, we can show Fredholmness. For functions on the base space these estimates are well known, and hence we need to study functions on the fiber separately. We now define this decomposition explicitly.

**Definition 4.21.** *For any continuous function $u$ on $P'_\infty$ define the projections*

$$u_b(x,t) = \frac{1}{2\pi} \int_{\pi^{-1}(x)} u\, \eta$$

*and*

$$u_f = u - u_b.$$

*A continuous function $u$ is called $S^1$ invariant if $u = u_b$. The function $u$ is called $S^1$ non-invariant if $u = u_f$. The operators that map $u$ to $u_b$ and $u_f$ will be denoted as $\pi_b$ and $\pi_f$ respectively.*

By construction, the space of continuous functions on $P'_\infty$ has a direct sum decomposition into $S^1$ invariant and $S^1$ non-invariant functions. Related to this splitting there are three analytical properties which turn out to be useful.

**Lemma 4.22.** *The operators $L_\delta$ and $\pi_b$ commute.*

*Proof.* Let $u \in C^{2,\alpha}_{cf}$ on a local trivialisation of $P_\infty$ and consider the Fourier series of $u$. Using the $S^1$ invariance of the metric and the various weight functions, one can show that the Laplacian acts diagonally over this Fourier decomposition. Therefore, it must commute with $\pi_b$. □

**Lemma 4.23.** *On any $S^1$ invariant domain $U$, the operators*

$$\begin{aligned}\pi_b &\colon C^0(U) \to C^0(U)\\ \pi_b &\colon C^{0,\alpha}_{cf}(U) \to C^{0,\alpha}_{cf}(U)\\ \pi_b &\colon \tilde{L}^2(U) \to \tilde{L}^2(U)\end{aligned}$$

*are bounded. The same holds for $\pi_f$.*

*Proof.* For any $u \in C^0(U)$ and $x \in U$,

$$|u_b(x)| \leq \frac{1}{2\pi}\int_0^{2\pi} |u(e^{it}\cdot x)| \,\mathrm{d}\,t \leq \frac{1}{2\pi}\int_0^{2\pi} \|u\|_{C^0(U)} \,\mathrm{d}\,t \leq \|u\|_{C^0(U)}.$$

This implies the boundedness in the $C^0$ case. When $u \in C^{0,\alpha}_{cf}(U)$, then for all $x, y \in U$ sufficiently close to each other,

$$-\|u\|_{C^{0,\alpha}_{cf}(U)} \leq \frac{u(x) - u(y)}{d(x,y)^\alpha} \leq \|u\|_{C^{0,\alpha}_{cf}(U)}.$$

By the $S^1$ invariance of the metric

$$-\|u\|_{C^{0,\alpha}_{cf}(U)} \leq \frac{u(e^{it}\cdot x)-u(e^{it}\cdot y)}{d(e^{it}\cdot x, e^{it}\cdot y)^\alpha} = \frac{u(e^{it}\cdot x)-u(e^{it}\cdot y)}{d(x,y)^\alpha} \leq \|u\|_{C^{0,\alpha}_{cf}(U)}.$$

for all $t\in\mathbb{R}$. Integrating this expression over $t$ yields

$$-\|u\|_{C^{0,\alpha}_{cf}(U)} \leq \frac{1}{2\pi}\int_0^{2\pi}\frac{u(e^{it}\cdot x)-u(e^{it}\cdot y)}{d(x,y)^\alpha}\,\mathrm{d}\,t = \frac{u_b(x)-u_b(y)}{d(x,y)^\alpha} \leq \|u\|_{C^{0,\alpha}_{cf}(U)}.$$

From this we conclude $\|u_b\|_{C^{0,\alpha}_{cf}(U)} \leq \|u\|_{C^{0,\alpha}_{cf}(U)}$. Finally, notice that in a local trivialisation $\pi_b$ projects fiberwise to one of the Fourier modes of $S^1$. Because the Fourier modes form an orthonormal basis, this is a bounded operator on $L^2$. □

**Proposition 4.24.** *Let $x\in P'_\infty$ and denote the orbit of $x$ as $S^1\cdot\{x\}$. For any continuous function $u$ that satisfies $u=u_f$,*

$$\begin{aligned}\|u\|_{C^0_{cf}(S^1\cdot\{x\})} \leq& 2\pi\frac{\epsilon\,\Omega}{\sqrt{h_\epsilon}}\|\,\mathrm{d}\,u\|_{C^0_{cf}(S^1\cdot\{x\})}\\ \|u\|_{L^2(S^1\cdot\{x\})} \leq& \frac{\epsilon\,\Omega}{\sqrt{h_\epsilon}}\|\,\mathrm{d}\,u\|_{\tilde{L}^2(S^1\cdot\{x\})}.\end{aligned}$$

*Specifically, when $B=\mathbb{R}^3$ or $B=\mathbb{R}^2\times S^1$, these estimates simplify to*

$$\begin{aligned}\|u\|_{C^0_{cf}(S^1\cdot\{x\})} \leq& 2\pi\frac{\epsilon}{e^\rho h_\epsilon}\|\,\mathrm{d}\,u\|_{C^0_{cf}(S^1\cdot\{x\})}\\ \|u\|_{L^2(S^1\cdot\{x\})} \leq& \frac{\epsilon}{e^\rho h_\epsilon}\|\,\mathrm{d}\,u\|_{L^2(S^1\cdot\{x\})},\end{aligned}$$

*and when $B=\mathbb{R}\times T^2$, they simplify to*

$$\begin{aligned}\|u\|_{C^0_{cf}(S^1\cdot\{x\})} \leq& 2\pi\frac{\epsilon}{h_\epsilon}\|\,\mathrm{d}\,u\|_{C^0_{cf}(S^1\cdot\{x\})}\\ \|u\|_{L^2(S^1\cdot\{x\})} \leq& \frac{\epsilon}{h_\epsilon}\|\,\mathrm{d}\,u\|_{L^2(S^1\cdot\{x\})}.\end{aligned}$$

*Proof.* Let $(x,t)\in P'_\infty$. Because $u$ is $S^1$ non-invariant, there exists a $t_0\in S^1$ such that $u(x,t_0)=0$. By the fundamental theorem of calculus,

$$u(x,t)=\int_{t_0}^{t}\frac{\partial u}{\partial t}\,\mathrm{d}\,t.$$

From definition 4.7 we estimate the circle radius, and hence

$$u(x,t) \leq \int_{t_0}^{t} \|\,\mathrm{d}\,u\|_{cf} \cdot \|\partial_t\|_{cf}\,\mathrm{d}\,t \leq 2\pi \frac{\epsilon\,\Omega}{\sqrt{h_\epsilon}} \|\,\mathrm{d}\,u\|_{C^0_{cf}(S^1\cdot\{x\})}.$$

In order to find the $L^2$ estimate, write $u(x,t) = \sum_n u_n(x) e^{int}$ and note that

$$\begin{aligned} \|u\|^2_{\tilde{L}^2(S^1\cdot\{x\})} &= \int |u|^2\,\mathrm{d}\,t = \sum_{n=1}^{\infty} u_n^2 \\ &\leq \sum_{n=0}^{\infty} n^2 u_n^2 \leq \|\,\mathrm{d}\,u(\partial_t)\|^2_{\tilde{L}^2(S^1\cdot\{x\})}. \end{aligned}$$

Therefore,

$$\|u\|^2_{\tilde{L}^2(S^1\cdot\{x\})} \leq \|\,\mathrm{d}\,u\|^2_{\tilde{L}^2(S^1\cdot\{x\})} \cdot \|\partial_t\|^2_{C^0(S^1\cdot\{x\})}.$$

□

Using the Poincaré inequality we are able to prove the Fredholmness of $\Delta^{GH}$. A crucial fact we need is that the circle fiber collapses at infinity. This is true for all cases except when $P_\infty$ is a trivial circle bundle. Hence we treat this case separately.

**Theorem 4.25.** *Assume that $B = \mathbb{R}^3$ or $B = \mathbb{R}^2 \times S^1$. Fix $\delta \in \mathbb{R} \setminus \mathbb{Z}$. There exist some uniform $0 < R_1 < R_0$ and $C > 0$ such that for any $u \in W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2)$ or $u \in C^{2,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)$,*

$$\begin{aligned} \|u\|_{W^{2,2}_\delta(P_\infty/\mathbb{Z}_2)} \leq & C\left[\|\Omega^{-2}\Delta^{GH}u\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)}\right] \quad \textit{or} \\ \|u\|_{C^{2,\alpha}_\delta(P_\infty/\mathbb{Z}_2)} \leq & C\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{C^0_\delta(K_\infty/\mathbb{Z}_2)}\right] \quad \textit{respectively.} \end{aligned}$$

*When $u$ vanishes on $\partial P'_\infty/\mathbb{Z}_2$,*

$$\begin{aligned} \|u\|_{W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2)} \leq & C\left[\|\Omega^{-2}\Delta^{GH}u\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)}\right] \quad \textit{or} \\ \|u\|_{C^{2,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} \leq & C\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{C^0_\delta(K_\infty/\mathbb{Z}_2)}\right] \quad \textit{respectively.} \end{aligned}$$

*Proof. (N.B. In this proof the value of $C$ will change from line to line.)* Assume without loss of generality that $B = \mathbb{R}^3$ and $u \in W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2)$. Consider the case

$u = u_f$. From Theorem 4.19,

$$\begin{aligned}\|u_f\|_{W^{2,2}_\delta(P_\infty/\mathbb{Z}_2)} \leq & C\left[\|\Omega^{-2}\Delta^{GH}u_f\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|u_f\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)}\right] \\ \leq & C\left[\|\Omega^{-2}\Delta^{GH}u_f\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|u_f\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)} + \|u_f\|_{L^2_\delta(P_\infty/\mathbb{Z}_2)}\right].\end{aligned}$$

Using the Poincaré inequality, we rewrite this as

$$\begin{aligned}\|u_f\|_{W^{2,2}_\delta(P_\infty/\mathbb{Z}_2)} \leq & C\Bigg[\|\Omega^{-2}\Delta^{GH}u_f\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|u_f\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)} \\ & + \frac{\epsilon}{e^{-R_1}\cdot\min_{P_\infty} h_\epsilon}\|\,\mathrm{d}\,u_f\|_{L^2_\delta(P_\infty/\mathbb{Z}_2)}\Bigg].\end{aligned}$$

If we pick $R_1$ such that $\frac{\epsilon C}{e^{-R_1}\cdot\min_{P_\infty} h_\epsilon} < \frac{1}{2}$, we conclude that

$$\frac{1}{2}\|u_f\|_{W^{2,2}_\delta(P_\infty/\mathbb{Z}_2)} \leq C\left[\|\Omega^{-2}\Delta^{GH}u_f\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|u_f\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)}\right].$$

Secondly, consider the case $u = u_b$. For $S^1$ invariant functions, $\Omega^{-2}\Delta^{GH}$ reduces to the standard Laplacian $\Delta^B$ on the basespace. As explained in the introduction of this chapter, the operator $\Delta^B$ is Fredholm in the norms given by Bartnik (1986) when $\delta \notin \mathbb{Z}$. By Lemma 4.8, these norms are equivalent to the Sobolev norms introduced in Definition 4.14. Therefore, there exists a uniform constant $C > 0$ independent of $u$ such that

$$\|u_b\|_{W^{2,2}_\delta(P_\infty/\mathbb{Z}_2)} \leq C\left[\|\Omega^{-2}\Delta^{GH}u_b\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|u_b\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)}\right].$$

Finally, consider the general case $u = u_b + u_f$. Combining the above estimates yields

$$\begin{aligned}\|u\|_{W^{2,2}_\delta(P_\infty/\mathbb{Z}_2)} \leq & \|u_b\|_{W^{2,2}_\delta(P_\infty/\mathbb{Z}_2)} + \|u_f\|_{W^{2,2}_\delta(P_\infty/\mathbb{Z}_2)} \\ \leq & C\Bigg[\|\Omega^{-2}\Delta^{GH}u_b\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|\Omega^{-2}\Delta^{GH}u_f\|_{L^22_\delta(P'_\infty/\mathbb{Z}_2)} \\ & + \|u_b\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)} + \|u_f\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)}\Bigg].\end{aligned}$$

Using Lemma 4.22 and 4.23 we get

$$\|u\|_{W^{2,2}_\delta(P_\infty/\mathbb{Z}_2)} \leq 2C\left[\|\Omega^{-2}\Delta^{GH}u\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)}\right],$$

which concludes the first estimate. The second estimate follows by a similar argument. □

In the above proof we picked $P_\infty$ such that $\frac{\epsilon C}{\min_{P_\infty} e^{-\rho h_\epsilon}} < \frac{1}{2}$ for all $0 < \epsilon < 1$. If we want to use the same proof when $B = \mathbb{R} \times T^2$, we need to show $\frac{\epsilon C}{\min_{P_\infty} h_\epsilon} < \frac{1}{2}$. However, when $P_\infty$ is the trivial circle bundle, the function $h_\epsilon$ is bounded above (see Lemma 3.4), and if $C$ is sufficient large we arrive at a contradiction. However, by forcing $\epsilon$ sufficient small, we still can require $\frac{\epsilon C}{\min_{P_\infty} h_\epsilon} < \frac{1}{2}$ and conclude:

**Theorem 4.26.** *Assume that $B = \mathbb{R} \times T^2$. Let $C$ as defined in Theorem 4.19 and assume that $\epsilon < \frac{\min_{P_\infty} h_\epsilon}{4\pi C}$. Fix $\delta \in \mathbb{R} \setminus \mathbb{Z}$ and $\alpha \in (0,1)$. There exists a constant $\tilde{C} > 0$, such that for any $u \in W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2)$ or $u \in C^{2,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)$,*

$$\begin{aligned}
\|u\|_{W^{2,2}_\delta(P_\infty/\mathbb{Z}_2)} \leq& \tilde{C}\left[\|\Omega^{-2}\Delta^{GH}u\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)}\right] \quad \textit{or} \\
\|u\|_{C^{2,\alpha}_\delta(P_\infty/\mathbb{Z}_2)} \leq& \tilde{C}\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{C^0_\delta(K_\infty/\mathbb{Z}_2)}\right] \quad \textit{respectively.}
\end{aligned}$$

*When $u$ vanishes on $\partial P'_\infty/\mathbb{Z}_2$,*

$$\begin{aligned}
\|u\|_{W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2)} \leq& \tilde{C}\left[\|\Omega^{-2}\Delta^{GH}u\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{L^2_\delta(K_\infty/\mathbb{Z}_2)}\right] \quad \textit{or} \\
\|u\|_{C^{2,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} \leq& \tilde{C}\left[\|\Omega^{-2}\Delta^{GH}u\|_{C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + \|u\|_{C^0_\delta(K_\infty/\mathbb{Z}_2)}\right] \quad \textit{respectively.}
\end{aligned}$$

**Corollary 4.27.** *Let $W^{2,2}_{\delta,0}(P'_\infty/\mathbb{Z}_2)$ be the space of all $W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2)$ functions that satisfy $u|_{\partial P'_\infty/\mathbb{Z}_2} = 0$. Under the conditions described in Theorem 4.25 or 4.26, the operator*

$$\Omega^{-2}\Delta^{GH} \colon W^{2,2}_{\delta,0}(P'_\infty/\mathbb{Z}_2) \to L^2_\delta(P'_\infty/\mathbb{Z}_2)$$

*is Fredholm.*

*Proof.* The proof is identical to the proof of Theorems 9.1.1 and 9.2.1 in Pacard (2008). His argument goes as follows:

To show that the kernel is finite dimensional, one assumes the contrary and considers an infinite sequence of orthonormal elements in the kernel. Using Rellich's compactness theorem one can find a subsequence that converges in $L^2_{\delta,0}(K_\infty/\mathbb{Z}_2)$. Using Theorem 4.25 or 4.26 one can show that this subsequence actually converges in $W^{2,2}_{\delta,0}(P'_\infty/\mathbb{Z}_2)$. Because each element in this sequence has norm one, the limit must also have norm one. At the same time, the limit is orthonormal to each element in the sequence and so the norm of the limit must be zero. This is a contradiction.

To show that the range is closed, one considers a sequence $u_i \in W^{2,2}_{\delta,0}(P'_\infty/\mathbb{Z}_2)$ such that $\Omega^{-2}\Delta^{GH}u_i$ converges to some $f \in L^2_\delta(P'_\infty/\mathbb{Z}_2)$. It is sufficient to show $u_i$ is bounded in $L^2_\delta(P'_\infty/\mathbb{Z}_2)$, because in that case Rellich's compactness theorem and Theorem 4.25 or 4.26 imply there is a subsequence converging to $u \in W^{2,2}_{\delta,0}(P'_\infty/\mathbb{Z}_2)$ such that $\Omega^{-2}\Delta u = f$. To show boundedness, one assumes the opposite and considers the normalised sequence $\hat{u}_i = u_i/\|u_i\|_{L^2_\delta(P'_\infty/\mathbb{Z}_2)}$. Using Rellich's compactness theorem and Theorem 4.25 or 4.26 again, one can show there is a converging subsequence in the kernel of $\Omega^{-2}\Delta$. This limit can be chosen orthogonal to all elements in the kernel and hence it is zero. At the same time its norm is equal to one. $\square$

## 4.4 Fredholm theory for the Laplacian

Knowing that the Laplacian is Fredholm, we study the kernel and co-kernel of $\Omega^{-2}\Delta^{GH}$ on $P'_\infty/\mathbb{Z}_2$ with Dirichlet boundary conditions. First, we consider the kernel of $\Omega^{-2}\Delta^{GH}$. In the case $\delta < 0$ injectivity follows from the maximum principle. In the next proposition we get a better result for $S^1$ non-invariant functions.

**Proposition 4.28.** *There exist uniform $R_1 > 0$ and $\tilde{\delta} > 0$, such that for any $\delta < \tilde{\delta}$ and $\alpha \in (0,1)$ there is no non-zero $u \in C^{2,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)$ that satisfies*

$$\begin{aligned} &u \text{ is } S^1 \text{ non-invariant},\\ &\Delta^{GH}u = 0, \text{ and}\\ &u|_{\partial P'/\mathbb{Z}_2} = 0. \end{aligned}$$

*Proof.* Let $R > R_1$. Using the notation as in Definition 4.6 consider the set

$$U_r = \pi^{-1}([R_1, r] \times \Sigma).$$

Using integration by parts, one can show that for any harmonic function $u$ on $U_r$ and $\delta \in \mathbb{R}$,

$$\|\mathrm{d}(e^{-2\delta\rho}u)\|^2_{L^2_{GH}(U_r)} = \int_{\partial U_r} e^{-4\delta\rho}u \, *^{GH} \mathrm{d}\, u + 4\delta^2 \cdot \|e^{-2\delta\rho}u \, \mathrm{d}\rho\|^2_{L^2_{GH}(U_r)}.$$

With respect to $g^{GH}$, the norm of $\mathrm{d}\,\rho$ is $\frac{1}{\sqrt{h_\epsilon}}$ or $\frac{1}{r\sqrt{h_\epsilon}}$ when $B = \mathbb{R}\times T^2$ or $B \neq \mathbb{R}\times T^2$ respectively. In any case this is bounded by one, but when $B \neq \mathbb{R}\times T^2$ one can make this term arbitrary small by changing $R_1$. By the Poincaré inequality, $\|e^{-2\delta\rho}u\|^2_{L^2_{GH}(U_r)} \le 2\pi\epsilon \cdot \|\,\mathrm{d}(e^{-2\delta\rho}u)\|^2_{L^2_{GH}(U_r)}$ and hence

$$(1 - 8\pi\epsilon\delta^2 \cdot \|\,\mathrm{d}\,\rho\|^2_{C^0_{GH}(P'_\infty/\mathbb{Z}_2)}) \cdot \|\,\mathrm{d}(e^{-2\delta\rho}u)\|^2_{L^2_{GH}(U_r)} \le \int_{\partial U_r} e^{-4\delta\rho}u \ *^{GH}\mathrm{d}\,u.$$

We pick $R_1$ and $\delta$ such that $8\pi\epsilon\delta^2 \cdot \|\,\mathrm{d}\,\rho\|^2_{C^0_{GH}(P'_\infty/\mathbb{Z}_2)} < 1$.

Finally we use the fact that $u$ vanishes on $\partial P'/\mathbb{Z}_2$. We are left with

$$\int_{\partial U_r} e^{-4\delta\rho}u \ *^{GH}\mathrm{d}\,u = \begin{cases} \epsilon e^{(1-4\delta)R}u(R)\frac{\partial u}{\partial\rho}(R)\ \int_\Sigma \mathrm{Vol}_{S^2}\wedge \mathrm{d}\,t & \text{if } B = \mathbb{R}^3 \\ \epsilon e^{-4\delta R}u(R)\frac{\partial u}{\partial\rho}(R)\int_\Sigma \mathrm{Vol}_{S^1\times S^1}\wedge \mathrm{d}\,t & \text{if } B = \mathbb{R}^2\times S^1 \\ \epsilon\, e^{-4\delta R}\ u(R)\frac{\partial u}{\partial\rho}(R)\ \int_\Sigma \mathrm{Vol}_{T^2}\wedge\mathrm{d}\,t & \text{otherwise.} \end{cases}$$

When $u \in C^{k,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)$, then there is a constant $C > 0$ such that

$$\int_{\partial U_r} e^{-4\delta\rho}u \ *^{GH}\mathrm{d}\,u \le C\cdot \begin{cases} e^{(1-2\delta)R} & \text{if } B = \mathbb{R}^3 \\ e^{-2\delta R} & \text{otherwise.} \end{cases}$$

This vanishes at infinity when $\delta > 0$ or when $\delta > \frac{1}{2}$. This implies that in the limit $r \to \infty$, $\|\,\mathrm{d}(e^{-2\delta\rho}u)\|^2_{L^2_{GH}(U_r)} = 0$ and hence $u$ must be a multiple of $e^{\delta\rho}$. The only $S^1$ non-invariant function that satisfies this is the constant zero function. □

**Corollary 4.29.** *There exist uniform $R_1 > 0$ and $\tilde\delta > 0$, such that for any $\delta < \tilde\delta$ and $\alpha \in (0,1)$ there is no non-zero $u \in W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2)$ that satisfies*

$$u \text{ is } S^1 \text{ non-invariant},$$
$$\Delta^{GH}u = 0, \text{ and}$$
$$u|_{\partial P'/\mathbb{Z}_2} = 0.$$

*Proof.* Using Theorem 4.18 and Theorem 4.20, $u \in C^{2,\alpha}$ and we apply the result from the Hölder case. □

One can solve the Laplace equation on $\mathbb{R}^2 \times S^1$ and $\mathbb{R} \times T^2$ using a Fourier decomposition. A small calculation will show that only the zeroth Fourier component will contribute when $\delta < 1$. Therefore, we only need to study the Laplace equation on $\mathbb{R}^2$ and $\mathbb{R}$. These harmonic functions are well known and as a reminder we will give them in the following Lemma:

**Proposition 4.30.** *Any $u \in C^{2,\alpha}_{\delta}(P'/\mathbb{Z}_2)$ or $u \in W^{2,2}_{\delta}(P'/\mathbb{Z}_2)$ that satisfies*

$$\begin{aligned} &u \text{ is } S^1 \text{ invariant,} \\ &\Omega^{-2}\Delta^{GH} u = 0, \text{ and} \\ &u|_{\partial P'/\mathbb{Z}_2} = 0, \end{aligned}$$

*will vanish when $\delta < 0$. For $\delta \in (0,1)$, $u$ must be of the form*

$$u = \begin{cases} \lambda + \mu \cdot e^{-\rho} & \text{if } B = \mathbb{R}^3, \\ \lambda + \mu \cdot \rho & \text{otherwise,} \end{cases}$$

*where $\lambda, \mu \in \mathbb{R}$ are chosen such that $u|_{\partial P'/\mathbb{Z}_2} = 0$.*

Next we focus on the cokernel. When one uses Sobolev spaces, one can calculate the cokernel of an operator by studying the kernel of its formal adjoint. Because the Laplacian is self-adjoint, we expect the formal adjoint of $L_\delta$ to be similar to itself. In the next proposition we make this precise. Combining this with our knowledge of the kernel from Proposition 4.30 we will get an explicit description of the range.

**Proposition 4.31.** *Let $W^{2,2}_{\delta,0}(P'_\infty/\mathbb{Z}_2)$ be the space of all $W^{2,2}_{\delta}(P'_\infty/\mathbb{Z}_2)$ functions that satisfy $u|_{\partial P'_\infty/\mathbb{Z}_2} = 0$. Under the conditions described in Theorem 4.25 or 4.26, $f \in L^2_\delta(P'_\infty/\mathbb{Z}_2)$ lies in the image of*

$$\Omega^{-2}\Delta^{GH} \colon W^{2,2}_{\delta,0}(P'_\infty/\mathbb{Z}_2) \to L^2_\delta(P'_\infty/\mathbb{Z}_2)$$

*if and only if $\langle f, e^\rho \cdot v\rangle_{\tilde{L}^2(P'_\infty/\mathbb{Z}_2)} = 0$ (or $\langle f, v\rangle_{\tilde{L}^2(P'_\infty/\mathbb{Z}_2)} = 0$ when $B \neq \mathbb{R}^3$) for all*

$$v \in \ker \Omega^{-2}\Delta^{GH} \colon \begin{cases} W^{2,2}_{-(\delta+1),0}(P'_\infty/\mathbb{Z}_2) \to L^2_{-(\delta+1)}(P'_\infty/\mathbb{Z}_2) & \text{if } B = \mathbb{R}^3 \\ W^{2,2}_{-\delta,0}(P'_\infty/\mathbb{Z}_2) \to L^2_{-\delta}(P'_\infty/\mathbb{Z}_2) & \text{otherwise.} \end{cases}$$

*Proof.* Because $L_\delta$ is Fredholm, we are left to calculate the formal adjoint of $L_\delta$. Recall from Definition 4.14 we equipped the Sobolev spaces with the volume form

$$\widetilde{\mathrm{Vol}} = \begin{cases} \mathrm{d}\,\rho \wedge \mathrm{Vol}_{S^2} \wedge \eta & \text{if } B = \mathbb{R}^3 \\ \mathrm{d}\,\rho \wedge \mathrm{Vol}_{T^2} \wedge \eta & \text{otherwise.} \end{cases}$$

Describing this using the volume form of $g^{GH}$, we get

$$\widetilde{\mathrm{Vol}} = \begin{cases} \epsilon^{-1}\,\Omega^2\, e^{-\rho}\,\mathrm{Vol}^{GH} & \text{if } B = \mathbb{R}^3 \\ \epsilon^{-1}\,\Omega^2\,\mathrm{Vol}^{GH} & \text{otherwise.} \end{cases}$$

Hence for all compactly supported functions $u$ and $v$,

$$\begin{aligned} \langle L_\delta u, v\rangle_{\tilde{L}^2(P'_\infty/\mathbb{Z}_2)} =& \langle e^{-\delta\rho}\Omega^{-2}\Delta^{GH}(e^{\delta\rho}u), v\rangle_{\tilde{L}^2(P'_\infty/\mathbb{Z}_2)} \\ =& \epsilon^{-1}\cdot \begin{cases} \langle e^{-(\delta+1)\rho}\Delta^{GH}(e^{\delta\rho}u), v\rangle_{L^2_{GH}(P'_\infty/\mathbb{Z}_2)} & \text{if } B = \mathbb{R}^3 \\ \langle e^{-\delta\rho}\Delta^{GH}(e^{\delta\rho}u), v\rangle_{L^2_{GH}(P'_\infty/\mathbb{Z}_2)} & \text{otherwise.} \end{cases} \end{aligned}$$

Using the self-adjointness of the Laplacian,

$$\langle L_\delta u, v\rangle_{\tilde{L}^2(P'_\infty/\mathbb{Z}_2)} = \begin{cases} \langle u, L_{-(\delta+1)}v\rangle_{\tilde{L}^2(P'_\infty/\mathbb{Z}_2)} & \text{if } B = \mathbb{R}^3 \\ \langle u, L_{-\delta}v\rangle_{\tilde{L}^2(P'_\infty/\mathbb{Z}_2)} & \text{otherwise.} \end{cases}$$

Therefore, the formal adjoint of $L_\delta$ is $L_{-(\delta+1)}$ or $L_{-\delta}$ respectively. This implies that $\tilde{f} \in \tilde{L}^2(P'_\infty/\mathbb{Z}_2)$ lies in the image of $L_\delta\colon \tilde{W}^{2,2}_0(P'_\infty/\mathbb{Z}_2) \to \tilde{L}^2(P'_\infty/\mathbb{Z}_2)$ if and only if $\langle \tilde{f}, \tilde{v}\rangle = 0$ for all $v \in \ker L_{-(\delta+1)}$ or $v \in \ker L_\delta$ respectively. Converting this to a condition on weighted spaces, we conclude $f = e^{\delta\rho}\tilde{f}$ lies in the image of $\Omega^{-2}\Delta^{GH}\colon W^{2,2}_{\delta,0} \to L^2_\delta$ if and only if $\langle f, e^\rho \cdot v\rangle = 0$ or $\langle f, v\rangle = 0$ for all

$$v = \begin{cases} e^{(\delta+1)\rho}\tilde{v} \in \ker \Omega^{-2}\Delta^{GH}\colon W^{2,2}_{-(\delta+1),0}(P'_\infty/\mathbb{Z}_2) \to L^2_{-(\delta+1)}(P'_\infty/\mathbb{Z}_2) & \text{if } B = \mathbb{R}^3 \\ e^{\delta\rho}\tilde{v} \in \ker \Omega^{-2}\Delta^{GH}\colon W^{2,2}_{-\delta,0}(P'_\infty/\mathbb{Z}_2) \to L^2_{-\delta}(P'_\infty/\mathbb{Z}_2) & \text{otherwise.} \end{cases}$$

□

When $B = \mathbb{R}^3$ the operator is injective for $\delta < 0$. According to Proposition 4.31, it must be surjective when $\delta > -1$. Hence it is an isomorphism for $\delta \in (-1, 0)$. However, when $B \neq \mathbb{R}^3$ there is no $\delta \in \mathbb{R}$ such that $\Omega^{-2}\Delta^{GH}$ is injective and surjective at the same time. Hence we need to manually enlarge the domain without adding new elements to the kernel. We claim that when $B \neq \mathbb{R}^3$, $\delta < 0$ and $|\delta| \ll 1$ the operator

$$\Omega^{-2}\Delta^{GH} \colon W^{2,2}_{\delta}(P'_{\infty}/\mathbb{Z}_2) \oplus \mathbb{R}\rho \to L^2_{\delta}(P'_{\infty}/\mathbb{Z}_2)$$

with Dirichlet boundary conditions is the isomorphism we are looking for. At first sight this only changes the kernel, because $\Delta^{GH}\rho = 0$, but the opposite is true. Namely, the boundary condition $u|_{\partial P'_{\infty}/\mathbb{Z}_2} = 0$ changes to $(u + \lambda\rho)|_{\partial P'_{\infty}/\mathbb{Z}_2} = 0$, which allows $u$ to be non-zero at the boundary. Hence we actually study the operator $\Omega^{-2}\Delta^{GH} \colon W^{2,2}_{\delta}(P'_{\infty}/\mathbb{Z}_2) \to L^2_{\delta}(P'_{\infty}/\mathbb{Z}_2)$ for functions that are constant on the boundary and we claim that this operator has an trivial co-kernel. In the following proof, we will reparametrise $W^{2,2}_{\delta}(P'_{\infty}/\mathbb{Z}_2) \oplus \mathbb{R}\rho$ as $W^{2,2}_{\delta,0}(P'_{\infty}/\mathbb{Z}_2) \oplus \mathbb{R}\phi$ for some function $\phi$ and show we have enough degrees of freedom to satisfy the conditions of Proposition 4.31.

**Theorem 4.32.** *Let $\delta \in (-1, 0)$ with $|\delta|$ sufficiently small. For any $f \in L^2_{\delta}(P'_{\infty}/\mathbb{Z}_2)$ there exists a unique $u \in W^{2,2}_{\delta}(P'_{\infty}/\mathbb{Z}_2)$ or $u \in W^{2,2}_{\delta}(P'_{\infty}/\mathbb{Z}_2) \oplus \mathbb{R}\rho$ such that*

$$\begin{aligned} \Omega^{-2}\Delta^{GH}u =& f \\ u|_{\partial P'/\mathbb{Z}_2} =& 0 \end{aligned}$$

*when $B = \mathbb{R}^3$ or $B \neq \mathbb{R}^3$ respectively.*

*Proof.* We only prove the case $B \neq \mathbb{R}^3$. Let $u + \lambda\rho \in W^{2,2}_{\delta}(P'_{\infty}/\mathbb{Z}_2) \oplus \mathbb{R}$ such that $\Delta(u + \lambda\rho) = 0$ and $u + \lambda\rho|_{\partial P'} = 0$. By Corollary 4.29 and Proposition 4.30, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$u + \lambda\rho = \alpha + \beta\rho,$$

because $W^{2,2}_{\delta}(P'_{\infty}/\mathbb{Z}_2) \oplus \mathbb{R}\rho \subset W^{2,2}_{-\delta}(P'_{\infty}/\mathbb{Z}_2)$. Dividing both sides by $\rho$ and taking the limit $\rho \to \infty$,

$$\beta = \lim_{\rho\to\infty}(\alpha\rho^{-1} + \beta) = \lim_{\rho\to\infty}(u\rho^{-1} + \lambda) = \lambda.$$

This automatically implies $u = \alpha$. The only constant function that is part of $W^{2,2}_{\delta}(P'_{\mathbb{Z}_2})$ is the constant zero function and therefore $\alpha = 0$. The boundary condition forces $\beta = 0$. This proves the injectivity of $\Omega^{-2}\Delta^{GH}$.

Before we consider the range of $\Omega^{-2}\Delta^{GH}$, we need to set up the following: Let $\alpha \in \mathbb{R}$ be such that $\alpha+\rho$ vanishes on the boundary of $P'_\infty/\mathbb{Z}_2$. Let $\chi$ be a smooth bump function on $P'_\infty/\mathbb{Z}_2$ such that $\chi|_{\partial P'_\infty/\mathbb{Z}_2} = 1$ and assume that $\langle \Omega^{-2}\Delta^{GH}(\chi\rho), \alpha+\rho\rangle \neq 0$.

To prove surjectivity, consider $f \in L^2_\delta(P'_\infty/\mathbb{Z}_2)$. There exists a $\beta \in \mathbb{R}$ such that $\langle f+\beta\cdot\Omega^{-2}\Delta^{GH}(\chi\rho), \alpha+\rho\rangle = 0$. According to Proposition 4.30 $a+\rho$ spans the kernel of $\Omega^{-2}\Delta^{GH}\colon W^{2,2}_{-\delta}(P'_\infty/\mathbb{Z}_2) \to L^2_{-\delta}(P'_\infty/\mathbb{Z}_2)$. This enables us to use Proposition 4.31 to show the existence of some $\hat{u} \in W^{2,2}_{\delta,0}(P'_\infty/\mathbb{Z}_2)$ such that

$$\Omega^{-2}\Delta^{GH}(\hat{u}) = f + \beta \cdot \Omega^{-2}\Delta^{GH}(\chi\rho)$$

Because $\chi$ is compactly supported, $u := \hat{u} - \beta\cdot\chi\rho$ is an element of $W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2)$ and we claim $u + \beta\rho \in W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2) \oplus \mathbb{R}\rho$ is the solution we are looking for. Because $\Delta^{GH}\rho = 0$,

$$\Omega^{-2}\Delta^{GH}(u+\beta\rho) = \Omega^{-2}\Delta^{GH}(\hat{u} - \beta\cdot\chi\rho) = f.$$

Using $\chi|_{\partial P'_\infty/\mathbb{Z}_2} = 1$ and $\hat{u}|_{\partial P'_\infty/\mathbb{Z}_2} = 0$,

$$u + \beta\rho|_{\partial P'_\infty/\mathbb{Z}_2} = \hat{u} + \beta(1-\chi)\rho|_{\partial P'_\infty/\mathbb{Z}_2} = 0.$$

This proves surjectivity. □

# 5 Perturbation to hyperkähler metrics

In this chapter we combine all our results and finally prove Theorem 1.1. Before we delve into the final part of this thesis, let us first recall what we have found in the previous chapters, and what the main objectives are.

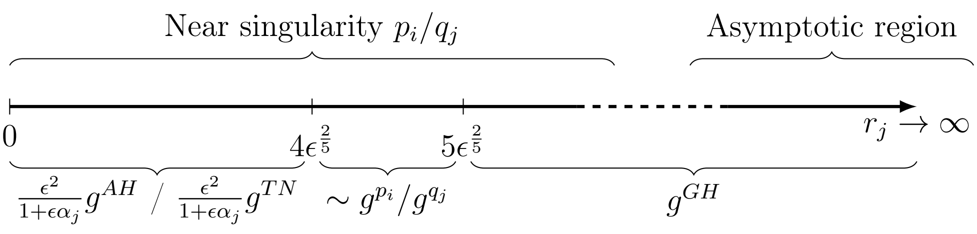



Figure 8: *Identifications of the approximate hyperkähler metric $g$ on the complete 4-dimensional manifold $M_{B,n}$.*

In Chapter 3, we constructed an almost hyperkähler manifold $M_{B,n}$ using a gluing construction. As shown in Figure 8, and proven in Theorem 3.23, the space $M_{B,n}$ has several distinct regions: For each non-fixed point $p_i$, there is a region with the rescaled Taub-NUT metric. For each fixed point $q_j$, there is a region with the rescaled metric of the Atiyah-Hitchin manifold. There is a bulk region that was constructed using the Gibbons-Hawking ansatz, and finally there are gluing regions which connect everything into a complete, connected, almost hyperkähler manifold.

Our main objective is to transform the approximate solution into a genuine gravitational instanton. In Section 2.2 we set the perturbation problem up, and explained how the hyperkähler condition can be phrased as an elliptic equation. Our goal is to solve this equation using the inverse function theorem. The main step is to understand its linearised version. We claimed at the end of Section 2.2, that this linearised version is an operator that is approximated by the Laplacian on functions.

To apply the inverse function theorem, we need the Laplacian to be an isomorphism and its inverse to be bounded uniformly with respect to the collapsing parameter $\epsilon$. For this we need to extend the asymptotic analysis of Chapter 4 to the whole of $M_{B,n}$. This will be the main study of the next four sections: In section 5.1 we extend our weighted norms and operators over the whole of $M_{B,n}$ and show that the weighted operator is uniformly elliptic. In Sections 5.2 and 5.3 we extend our elliptic estimates and we show that the Laplacian is Fredholm. Here we also determine the (co)-kernel and explain for which spaces the Laplacian is bijective. The whole of

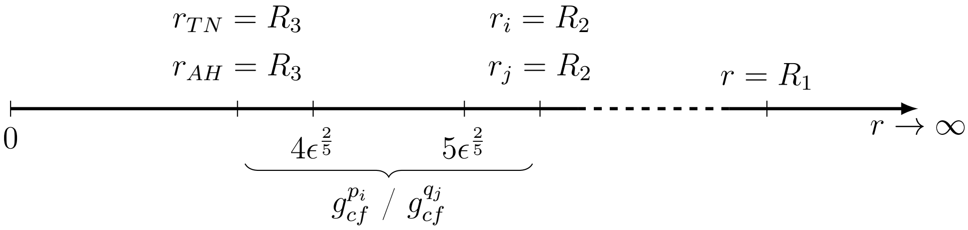
Section 5.4 will be devoted to the proof of the uniform bounded inverse estimate.

Finally, in section 5.5 we revisit the inverse function theorem and we prove all the claims we made in Section 2.2. Namely, we show that with the correct weighted norms, our approximate solution is indeed approximately hyperkähler. Secondly, we show that the linearized operator is a small perturbation of the Laplacian and finally we show that the non-linear part of our error estimate acts ‘quadratically’. This enables us to apply the inverse function theorem and prove the main result.

## 5.1 Global metric

In Chapter 4 we considered a conformally rescaled metric $g_{cf} = \Omega^2 g^{GH}$ and the operator $\Omega^{-2}\Delta^{GH}$ on the asymptotic region of $M_{B,n}$. We introduced a radial parameter $\rho$, and defined the weighted operator $L_\delta := e^{-\delta\rho}\Omega^{-2}\Delta^{GH}(e^{\delta\rho}\ldots)$. We showed that, with respect to $g_{cf}$, the operator $L_\delta$ is elliptic. By analysing $g_{cf}$, we found regularity estimates for $L_\delta$, and with these estimates we showed $L_\delta$ is Fredholm. We determined the (co)-kernel of this operator and, by extending the domain with a function $\phi$, we concluded $L_\delta$ is an isomorphism for $\delta < 0$ and $|\delta|$ sufficiently small.

The functions $\Omega$, $\rho$ and $\phi$ were defined on the asymptotic region of our manifold and in this section we will extend these functions to the interior. This choice cannot be arbitrary if we want to show the existence of a uniformly bounded inverse. Our choices are summarised in Definition 5.1, but before that we give a heuristic argument for each of them.

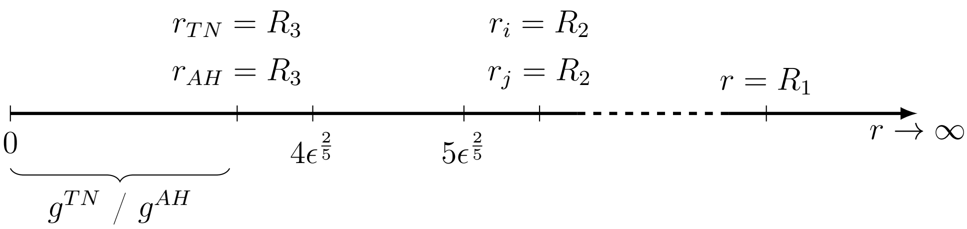



First we consider the metric near the gluing region. Up to double cover and depending on the kind of singularity, the metric $g$ approximates the model metrics

$$\begin{aligned} g^{p_i} :=& h_\epsilon^{p_i}\left(\mathrm{d}\, r_i^2 + r_i^2 g_{S^2}\right) + \frac{\epsilon^2}{h_\epsilon^{p_i}}(\eta^{p_i})^2, & h_\epsilon^{p_i} =& 1 + \epsilon(\alpha_i + \frac{1}{2r_i}), \\ g^{q_j} :=& h_\epsilon^{q_j}\left(\mathrm{d}\, r_j^2 + r_j^2 g_{S^2}\right) + \frac{\epsilon^2}{h_\epsilon^{q_j}}(\eta^{q_j})^2, & h_\epsilon^{q_j} =& 1 + \epsilon(\alpha_j - \frac{2}{r_j}). \end{aligned}$$

Because we have our error estimate with respect to $g_{cf}^{p_i}$ and $g_{cf}^{q_j}$, we will do our analysis with respect to these metrics[21]. These conformally rescaled metrics are part of the metrics we studied in Chapter 4 and hence we can pick $\Omega$ and $\rho$ as we defined before, i.e. $\Omega := r_i^{-1}(h_\epsilon^{p_i})^{-\frac{1}{2}}$ or $\Omega := r_j^{-1}(h_\epsilon^{q_j})^{-\frac{1}{2}}$ and $\rho := \log r_i$ or $\rho := \log r_j$ respectively. The function $\phi$ is only needed for the analysis on the asymptotic geometry and so we pick $\phi := 0$ on this region.

There is another reason why we measure this part with respect to the model metrics. Namely, there are two ways to view the complete manifold $M_{B,n}$. Normally we view $(M_{B,n}, g)$ as a fixed manifold where the circle fibers decay and very small regions are replaced by the Atiyah-Hitchin manifold and Taub-NUT spaces. Alternatively, if we conformally rescale by $\sim \frac{1}{\epsilon^2}$, we can view it as fixed Atiyah-Hitchin manifolds and Taub-NUT spaces and the gluing is done $\epsilon^{-1}$ far away. For this second picture we use that $g^{AH}$ is approximately the Taub-NUT metric $g^{TN'}$ with mass $-4$. To use our asymptotic analysis from before, we want to measure our function spaces with respect to $g_{cf}^{TN'} := \frac{1}{r_{AH}^2(1-2r_{AH}^{-1})} g^{TN'}$. Using the identification $r_i = \frac{\epsilon}{1+\epsilon\alpha_j} r_{AH}$, one can prove $g_{cf}^{TN'} = g_{cf}^{q_j}$. Similarly, one can show $g_{cf}^{TN} = g_{cf}^{p_i}$.




Secondly, consider large, fixed, $\epsilon$-invariant, compact regions inside the Taub-NUT spaces and the Atiyah-Hitchin manifolds. In these regions $g = \frac{\epsilon^2}{1+\epsilon\alpha_i} g^{TN}$ or $g = \frac{\epsilon^2}{1+\epsilon\alpha_j} g^{AH}$ respectively. To make our estimates independent of $\epsilon$, we conformally rescale back to $g^{TN}$ and $g^{AH}$, and hence we want $\Omega$ to be $\frac{1}{\epsilon}\sqrt{1+\epsilon\alpha_i}$ or $\frac{1}{\epsilon}\sqrt{1+\epsilon\alpha_j}$ respectively. Because on compact sets all weighted norms are equivalent, we pick $\rho$ to be constant and again we pick $\phi = 0$.

Having chosen $g_{cf}$ on the bubbles, on the gluing regions and on the asymptotic part of $M_{B,n}$, we now interpolate these metrics. For this we keep two things in mind. First,

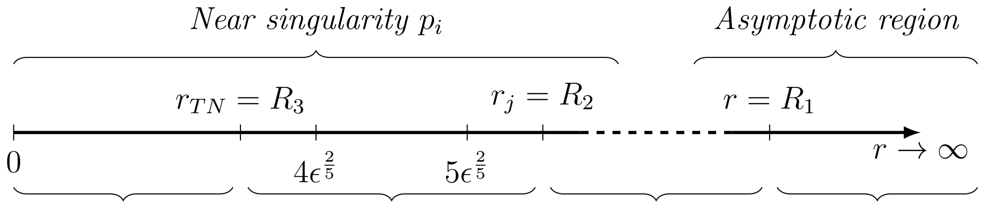
[21]Technically, this choice implies $g_{cf}$ is not a conformal rescaling of $g$, because $g$ only approximates the model metrics on the gluing region. To be consistent with the notation of Chapter 4, we still use $g_{cf}$ for this globally defined metric.

we pick the boundary of each region on places for which we have explicit control of the metric. For example, we interpolate $g_{AH}$ and $g_{cf}^{q_j}$ only in the asymptotic region of the Atiyah-Hitchin manifold, because we have the approximation $g^{AH} = g^{TN'} + \mathcal{O}(e^{-r_{AH}})$ only at infinity. Secondly, we want the transition between the regions to happen on fixed compact sets, so that we can practically ignore them in our analysis. Therefore, we define $g_{cf}$ as follows:

**Definition 5.1.** *Let $R_1 > 0$ be such that $P'_\infty := \pi^{-1}([R_1, \infty) \times \Sigma)$ is the asymptotic region described in Section 4.3. Pick $R_2, R_3 > 0$ such that $R_2 \ll 1$ and $R_3 \gg 1$. Consider $M_{B,n}$ as the disjoint union of the regions*

$$\begin{array}{ll} \{r_{TN} < R_3\}, & \{r_{AH} < R_3\}, \\ \{r_{TN} > R_3 \text{ and } r_i < R_2\}, & \{r_{AH} > R_3 \text{ and } r_j < R_2\}, \\ \{r_i, r_j > R_2 \text{ and } r < R_1\}, \text{ and} & \{r > R_1\}, \end{array}$$

*where $r_{TN}$, $r_{AH}$, $r_i$, $r_j$ and $r$ are the radial parameters induced by $g^{TN}$, $g^{AH}$, $g^{p_i}$, $g^{q_j}$ and $g^{GH}$ respectively. On the interior of each region, define the metric $g_{cf}$ and the functions $\Omega, \rho, \phi \in C^\infty(M_{B,n})$ as shown in the following tables and interpolate the metric and functions on the overlap:*

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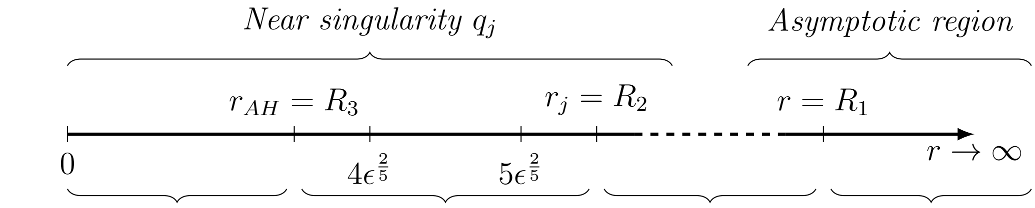
|---|---|---|---|---|
| $g_{cf}$ | $g^{TN}$ | $g_{cf}^{p_i} = g_{cf}^{TN}$ | $g^{GH}$ | $g_{cf}^{GH}$ |
| $\Omega$ | $\frac{\sqrt{1+\epsilon\alpha_i}}{\epsilon}$ | $r_i^{-1}(h_\epsilon^{p_i})^{-\frac{1}{2}}$ | 1 | *(Def. 4.7)* |
| $\rho$ | $\log\left(\frac{\epsilon}{1+\epsilon\alpha_i} R_3\right)$ | $\log(r_i)$ | 1 | *(Def. 4.9)* |
| $\phi$ | 0 | 0 | 0 | $\phi = \rho$ |




| | | | | |
|---|---|---|---|---|
| $g_{cf}$ | $g^{AH}$ | $g^{q_j}_{cf} = g^{TN'}_{cf}$ | $g^{GH}$ | $g^{GH}_{cf}$ |
| $\Omega$ | $\frac{\sqrt{1+\epsilon\alpha_j}}{\epsilon}$ | $r_j^{-1}(h^{q_j}_\epsilon)^{-\frac{1}{2}}$ | 1 | *(Def. 4.7)* |
| $\rho$ | $\log\left(\frac{\epsilon}{1+\epsilon\alpha_j}R_3\right)$ | $\log(r_j)$ | 1 | *(Def. 4.9)* |
| $\phi$ | 0 | 0 | 0 | $\phi = \rho$ |

With this choice of $g_{cf}$ we define Hölder and Sobolev norms. We only introduced the Sobolev norms to show bijectivity of the Laplacian. For this we don't need uniformity in $\epsilon$ and hence we can be less strict in our definition. However, this sloppy work will imply that some estimates are not uniform.

**Definition 5.2.** *Let $g_{cf}$, $\Omega$, $\rho$ and $\phi$ be as described in Definition 5.1. Let $\widetilde{\mathrm{Vol}}$ be the volume form chosen in Definition 4.14 and extend it to a global volume form on $M_{B,n}$.*

*For all $\delta \in \mathbb{R}$, we define the weighted operator $L_\delta$ as*

$$L_\delta = e^{-\delta\rho}\Omega^{-2}\Delta^g(e^{\delta\rho}\ldots).$$

*For any $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $\delta \in \mathbb{R}$, we define the weighted Hölder norm on $U \subseteq M_{B,n}$ as*

$$\|u\|_{C^{k,\alpha}_\delta(U)} = \|e^{-\delta\rho} \cdot u\|_{C^{k,\alpha}_{g_{cf}}(U)}.$$

*For any $k \in \mathbb{N}$, $\delta \in \mathbb{R}$, we define the weighted $L^2$ and Sobolev norm on $U \subseteq M_{B,n}$ as*

$$\begin{aligned}\langle u, v\rangle_{L^2_\delta(U)} &= \langle e^{-\delta\rho}\, u, e^{-\delta\rho}\, v\rangle_{\tilde{L}^2(U)}\\ \|u\|^2_{W^{k,2}_\delta(U)} &= \sum_{n=0}^{k} \|\, |\nabla^n(e^{-\delta\rho} \cdot u)|_{cf}\|^2_{\tilde{L}^2(U)},\end{aligned}$$

*where $\tilde{L}^2(U)$ is the $L^2$ norm with respect to the volume form $\widetilde{\mathrm{Vol}}$.*

With this metric and operator we start our global analysis. We first show that $\Omega^{-2}\Delta^g$ is elliptic and that this ellipticity is uniform in $\epsilon$. We will also show that it is uniform in $\delta$ if $\delta$ is restricted to a closed interval. This fact will be used when we convert our results from Sobolev to Hölder norms.

**Proposition 5.3.** *For all $k \in \mathbb{N}_{\geq 2}$, $\alpha \in (0,1)$, $\delta \in \mathbb{R}$, the operator $\Omega^{-2}\Delta^g \colon C^{k,\alpha}_\delta(M_{B,n}) \to C^{k-2,\alpha}_\delta(M_{B,n})$ is strictly elliptic in the sense of Proposition 4.10. This property is uniform in $\epsilon$. Moreover, if $\delta$ is restricted to a closed interval, then $\Omega^{-2}\Delta^g$ is strictly elliptic uniformly with respect to $\delta$ and $\epsilon$.*

*Proof.* First consider the weightless case, i.e. when $\delta = 0$. Later we generalise our result to $\delta \in \mathbb{R}$. We study this operator on each region separately. First consider the region inside the bubbles, i.e. where $r_{TN} < R_3$ or $r_{AH} < R_3$. Because $\Omega$ is constant, $\Omega^{-2}\Delta^g = \Delta^{g_{cf}}$. By definition, the Laplacian is strictly elliptic and hence $\Omega^{-2}\Delta^g$ is.

Next consider the compact region away from the singularities and away from infinity, i.e. when $r_i > R_2$, $r_j > R_2$ and $r < R_1$. For this part there is no rescaling, i.e. $\Omega^{-2}\Delta^g = \Delta^{g_{cf}}$, and ellipticity follows trivially. The ellipticity in the asymptotic case (where $r > R_1$) is already shown in Proposition 4.10.

Next consider the region near the gluing regions, i.e. when $r_{AH} > R_3$ and $r_j < R_2$. For this let $\{x_i\}$ be Riemann normal coordinates with respect to $g^{q_j}$. In these coordinates, $\Omega^{-2}\Delta^g$ is given by

$$\Omega^{-2}\Delta^g = \Omega^{-2}\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial u}{\partial x_j}\right).$$

The metric $g$ is approximated by $g = g^{q_j} + \mathcal{O}(\epsilon^3 r_j^{-1}) + \mathcal{O}(\epsilon r_j^4) = g^{q_j} + \mathcal{O}(\epsilon)$ and therefore

$$\begin{aligned}\Omega^{-2}\Delta^g =& \Omega^{-2}(1+\mathcal{O}(\epsilon))\frac{\partial}{\partial x_i}\left((\mathrm{Id} + \mathcal{O}(\epsilon))^{ij}\frac{\partial u}{\partial x_j}\right)\\ =& \Omega^{-2}\Delta^{g^{q_j}}u + \mathcal{O}(\epsilon)\cdot \|\operatorname{d} u\|_{C^{1,\alpha}_{cf}}.\end{aligned}$$

The model operator $\Omega^{-2}\Delta^{g^{q_j}}$ we already studied in Section 4.1, and by Proposition 4.10 the operator $\Omega^{-2}\Delta^{g^{q_j}}$ is uniformly strictly elliptic. Hence, this must also apply to $\Omega^{-2}\Delta^g$. A similar statement is true for the gluing regions near the non-fixed singularities $p_i$.

Finally, we study the boundary between the regions. Away from the singularities all interpolations happen on fixed compact sets between asymptotically similar metrics and all relevant functions are uniformly bounded with respect to $\epsilon$. Therefore, these interpolations will not change the ellipticity of $\Omega^{-2}\Delta^g$, and hence $\Omega^{-2}\Delta^g \colon C_0^{k,\alpha}(M_{B,n}) \to C_0^{k-2,\alpha}(M_{B,n})$ is strictly elliptic uniform in $\epsilon$.

In order to consider the case when $\delta \neq 0$, we just need to compare $L_\delta$ with $\Omega^{-2}\Delta^g$. Let $x_i$ Riemann normal coordinates of $g_{cf}$ and write

$$\Omega^{-2}\Delta^g = \alpha_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \beta_i \frac{\partial}{\partial x_i} + \gamma.$$

Then for all $u \in C^2(M_{B,n})$,

$$\begin{aligned} L_\delta\, u :=& e^{-\delta\rho}\Omega^{-2}\Delta^g(e^{\delta\rho}u) \\ =& \Omega^{-2}\Delta^g u + \delta\, \alpha_{ij}\left[\frac{\partial u}{\partial x_i}\frac{\partial \rho}{\partial x_j} + \frac{\partial \rho}{\partial x_i}\frac{\partial u}{\partial x_j}\right] + \delta \cdot \left[\delta\, \alpha_{ij}\frac{\partial^2 \rho}{\partial x_i \partial x_j} + \beta_i \frac{\partial \rho}{\partial x_i}\right] \cdot u. \end{aligned}$$

This will be uniformly strictly elliptic if $\mathrm{d}\,\rho$ and $\nabla\,\mathrm{d}\,\rho$ are uniformly bounded. This is true by construction. Moreover, the difference between $e^{-\delta\rho}\Omega^{-2}\Delta^g(e^{\delta\rho}u)$ and $\Omega^{-2}\Delta^g u$ is linearly dependent on $\delta$. Hence, if $\delta$ is fixed to a closed interval, $\Omega^{-2}\Delta$ is strictly elliptic, uniformly with respect to $\delta$ and $\epsilon$. □

## 5.2 Bijectivity in Sobolev spaces

Next we study the domain and range of $\Omega^{-2}\Delta^g$. In Section 4.4 this is already done in the asymptotic region of our manifold $M_{B,n}$, and we only need to extend these results globally. For now, we restrict ourself to Sobolev norms. In the next section we will focus on Hölder norms.

First we show that $\Omega^{-2}\Delta$ is Fredholm. For this we extend the regularity results from Section 4.3 to $M_{B,n}$. In the construction of $M_{B,n}$ we extended the bulk space $P_\infty/\mathbb{Z}_2$ by some compact set. Hence, to get global elliptic estimates, we combine the standard elliptic estimates on Riemannian manifold with the results from Section 4.3. Using this, we extend Theorems 4.19, 4.20, 4.25 and 4.26 without proof.

**Lemma 5.4.** *Let $\epsilon \in (0,1)$, $k \in \mathbb{N}_{\geq 2}$ and $\delta \in \mathbb{R}$. There exists a constant $C_\epsilon > 0$ such that for any $L^2_\delta$-bounded $u \in W^{k,2}_{loc}(M_{B,n})$ with $\Omega^{-2}\Delta^g u \in W^{k-2,2}_\delta(M_{B,n})$,*

$$u \in W^{k,2}_\delta(M_{B,n})$$

*and*

$$\|u\|_{W^{k,2}_\delta(M_{B,n})} \leq C_\epsilon \left[\|\Omega^{-2}\Delta^g u\|_{W^{k-2,2}_\delta(M_{B,n})} + \|u\|_{L^2_\delta(M_{B,n})}\right].$$

**Lemma 5.5.** *Let $\epsilon \in (0,1)$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$. There exists a constant $C_\epsilon > 0$ such that for any $u \in L^2_\delta(M_{B,n})$ with $\Omega^{-2}\Delta^g u \in C^{0,\alpha}_\delta(M_{B,n})$,*

$$u \in C^{2,\alpha}_\delta(M_{B,n})$$

*and*

$$\|u\|_{C^{2,\alpha}_\delta(M_{B,n})} \leq C_\epsilon \left[\|\Omega^{-2}\Delta^g u\|_{C^{0,\alpha}_\delta(M_{B,n})} + \|u\|_{L^2_\delta(M_{B,n})}\right].$$

**Lemma 5.6.** *Let $\epsilon \in (0,1)$. (When $B = \mathbb{R} \times T^2$, we also assume that $\epsilon$ is sufficiently small.) Fix $\delta \in \mathbb{R} \setminus \mathbb{Z}$. There exist a constant $C_\epsilon > 0$ and a compact set $K$, such that for any $u \in W^{2,2}_\delta(M_{B,n})$,*

$$\|u\|_{W^{2,2}_\delta(M_{B,n})} \leq C_\epsilon \left[\|\Omega^{-2}\Delta^g u\|_{L^2_\delta(M_{B,n})} + \|u\|_{L^2_\delta(K)}\right].$$

Lemma 5.6 implies that the operator $\Omega^{-2}\Delta^g$ is Fredholm. With this we can study the (co)-kernel explicitly. When $B = \mathbb{R}^3$, this is very straightforward due to the maximum principle:

**Theorem 5.7.** *Assume that $B = \mathbb{R}^3$. Let $\delta \in (-1,0)$ and $k \in \mathbb{N}_{\geq 2}$. For any $f \in W^{k-2,2}_\delta(M_{\mathbb{R}^3,n})$ there exists a unique $u \in W^{k,2}_\delta(M_{\mathbb{R}^3,n})$ such that*

$$\Omega^{-2}\Delta^g u = f.$$

*Proof.* By Lemma 5.4, it is sufficient to show $\Omega^{-2}\Delta^g \colon W^{2,2}_\delta(M_{\mathbb{R}^3,n}) \to L^2_\delta(M_{\mathbb{R}^3,n})$ is an isomorphism. Injectivity is trivial: Assume that $u$ is a harmonic function. Because $\delta < 0$, $u$ must be decaying. By the maximum principle $u$ is arbitrary small and hence $u = 0$. Therefore the kernel of $\Omega^{-2}\Delta^g$ is trivial.

Next we study the cokernel of $\Omega^{-2}\Delta^g \colon W^{2,2}_\delta(M_{\mathbb{R}^3,n}) \to L^2_\delta(M_{\mathbb{R}^3,n})$. As we did in chapter 4, this is equivalent in studying the cokernel of the weighted operator $L_\delta \colon W^{2,2}_{cf}(M_{\mathbb{R}^3,n}) \to L^2_{cf}(M_{\mathbb{R}^3,n})$. To find the formal adjoint, recall from Definition 5.2 that the $L^2$-space is measured with respect to some volume form $\widetilde{\mathrm{Vol}}$. Let $f$ be the smooth function such that $\widetilde{\mathrm{Vol}} = f^2 \mathrm{Vol}^g$. The formal adjoint of $L_\delta$ is given by

$$L^*_\delta = f^{-2} e^{\delta\rho} \Delta^g \left( \frac{f^2}{\Omega^2} e^{-\delta\rho} \dots \right),$$

which is $L_{-(1+\delta)}$ on the asymptotic part of $M_{\mathbb{R}^3,n}$. If $\delta > -1$, the maximum principle implies $L^*_\delta$ has a trivial kernel. Therefore, when $\delta \in (-1, 0)$, the operator $\Omega^{-2}\Delta^g$ is bijective. □

This argument fails when $B \neq \mathbb{R}^3$, because the formal adjoint of $L_\delta$ is $L_{-\delta}$ on the asymptotic region. However, as in Theorem 4.32, we expect that the operator $\Omega^{-2}\Delta^g \colon W^{k,2}_\delta(M_{B,n}) \oplus \mathbb{R}\,\phi \to W^{k-2,2}_\delta(M_{B,n})$ will be bijective. We will show this in three steps. First, we will give a preliminary result about the (co)-kernel of the Laplacian. Secondly, we will use our results about the Dirichlet boundary conditions to show surjectivity. Finally, we will prove injectivity.

**Lemma 5.8.** *Assume that $B \neq \mathbb{R}^3$. When $\delta < 0$ the operator $\Omega^{-2}\Delta^g \colon W^{2,2}_\delta(M_{B,n}) \to L^2_\delta(M_{B,n})$ is injective. When $\delta > 0$ the operator $\Omega^{-2}\Delta^g \colon W^{2,2}_\delta(M_{B,n}) \to L^2_\delta(M_{B,n})$ is surjective.*

*Proof.* This is the maximum principle applied on $\Omega^{-2}\Delta^g$ and on its formal adjoint. □

According to Lemma 5.8, there always exists an inverse, but this inverse might have the wrong decay rate. Using the Poisson equation on the asymptotic region we will show that this cannot happen.

**Theorem 5.9.** *Assume that $B \neq \mathbb{R}^3$. Let $\delta \in (-1, 0)$ with $|\delta|$ sufficiently small and $k \in \mathbb{N}_{\geq 2}$. For any $f \in W^{k-2,2}_\delta(M_{B,n})$ there exists a $u \in W^{k,2}_\delta(M_{B,n}) \oplus \mathbb{R}\phi$*

*such that*

$$\Omega^{-2}\Delta^g u = f.$$

*Proof.* By Lemma 5.4 it is sufficient to show that $\Omega^{-2}\Delta^g \colon W^{2,2}_\delta(M_{B,n}) \oplus \mathbb{R}\phi \to L^2_\delta(M_{B,n})$ is surjective. Let $f \in L^2_\delta(M_{B,n})$. By Lemma 5.8 there exists a $u \in W^{2,2}_{-\delta}(M_{B,n})$ such that

$$\Omega^{-2}\Delta u = f.$$

Our goal is to show that $u \in W^{2,2}_\delta(M_{B,n}) \oplus \mathbb{R}\phi$. Let $\chi$ be a small bump function on $M_{B,n}$ that is one on $\partial P'_\infty/\mathbb{Z}_2$. By Theorem 4.32 there exist a function $u_\infty \in W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2)$ and $\lambda \in \mathbb{R}$ such that

$$\begin{aligned}\Omega^{-2}\Delta^g(u_\infty + \lambda\phi) =& f - \Omega^{-2}\Delta^g(\chi u),\\ (u_\infty + \lambda\phi)|_{\partial P'/\mathbb{Z}_2} =& 0.\end{aligned}$$

The term $\Omega^{-2}\Delta^g(\chi u)$ is added, because it induces the conditions

$$\begin{aligned}\Omega^{-2}\Delta^g(u_\infty + \chi u + \lambda\phi) =& f,\\ (u_\infty + \chi u + \lambda\phi)|_{\partial P'/\mathbb{Z}_2} =& u.\end{aligned}$$

At the same time, the restriction of $u$ to the region $P'/\mathbb{Z}_2$ also satisfies

$$\begin{aligned}\Omega^{-2}\Delta^g(u) =& f\\ u|_{\partial P'/\mathbb{Z}_2} =& u,\end{aligned}$$

and hence $u_\infty + \chi u + \lambda\phi - u$ is a harmonic function on $P'/\mathbb{Z}_2$ with Dirichlet boundary conditions. Because $W^{2,2}_\delta \oplus \mathbb{R}\phi$ is a subset of $W^{2,2}_{-\delta}$, and the harmonics of $W^{2,2}_{-\delta}(P'_\infty/\mathbb{Z}_2)$ are known by Proposition 4.30,

$$u_\infty + \chi u + \lambda\phi - u = \alpha + \beta\rho$$

for some $\alpha, \beta \in \mathbb{R}$. From this we make two observations: First, $u + \alpha + (\beta - \lambda)\phi$ is an element of $W^{2,2}_{-\delta}(M_{B,n})$, and secondly, it is also equal to $u_\infty + \chi u \in W^{2,2}_\delta(P'_\infty/\mathbb{Z}_2)$. Because $M_{B,n}$ is the union of a compact set with $P'_\infty/\mathbb{Z}_2$ and all weighted $W^{2,2}$

norms on compact sets are equivalent,

$$u + \alpha + (\beta - \lambda)\phi \in W^{2,2}_\delta(M_{B,n}).$$

We conclude $u + \alpha \in W^{2,2}_\delta(M_{B,n}) \oplus \mathbb{R}\phi$ and $\Omega^{-2}\Delta^g(u+\alpha) = f$, which proves surjectivity. □

**Theorem 5.10.** *Assume that $B \neq \mathbb{R}^3$. Let $\delta \in (-1,0)$ with $|\delta|$ sufficiently small and $k \in \mathbb{N}_{\geq 2}$. The operator*

$$\Omega^{-2}\Delta^g \colon W^{k,2}_\delta(M_{B,n}) \oplus \mathbb{R}\phi \to L^2_\delta(M_{B,n})$$

*has a trivial kernel.*

*Proof.* Assume the contrary, and let $v$ be a non-zero element of $W^{2,2}_\delta(M_{B,n})$ and $\lambda \in \mathbb{R}$ such that $\Delta(v + \lambda\phi) = 0$. If $\lambda = 0$, Lemma 5.8 implies $v = 0$ and this contradicts our assumption. Therefore, we can rescale our harmonic function such that $\lambda = 1$.

We claim that our assumption implies surjectivity of $\Omega^{-2}\Delta^g \colon W^{2,2}_\delta(M_{B,n}) \to L^2_\delta(M_{B,n})$. Indeed, let $f \in L^2_\delta(M_{B,n})$. By Theorem 5.9 there must be a $u \in W^{2,2}_\delta(M_{B,n})$ and a $\lambda \in \mathbb{R}$ such that $\Omega^{-2}\Delta(u + \lambda\phi) = f$. By our choice of $v$, we also have

$$\Omega^{-2}\Delta^g(u - \lambda v) = \Omega^{-2}\Delta^g(u + \lambda\phi - \lambda(v + \phi)) = f.$$

Hence $u - \lambda v \in W^{2,2}_\delta(M_{B,n})$ is an inverse of $f$.

We claim that surjectivity of $\Omega^{-2}\Delta^g \colon W^{2,2}_\delta(M_{B,n}) \to L^2_\delta(M_{B,n})$ leads to a contradiction. Indeed, when $\Omega^{-2}\Delta^g$ is surjective, then $L_\delta$ is surjective and its formal adjoint must be injective. As shown in the proof of Theorem 5.7, the formal adjoint is $L_{-\delta}$ on the asymptotic part of $M_{B,n}$. Because $\delta < 0$, the constants are part of the kernel of $L^*_\delta$, but we just have shown that the kernel of $L^*_\delta$ is trivial. Therefore, $v$ does not exist. □

## 5.3 Bijectivity in Hölder spaces

Knowing that $\Omega^{-2}\Delta^g$ is an isomorphism in weighted Sobolev norms, we can ask whether the same result is true for Hölder norms. For this, we embed Hölder norms into Sobolev spaces. This almost works, but we will lose some weight. Namely, for any $\tilde{\delta} > \delta$ we have the embedding

$$C^0_\delta(M_{B,n}) \subset L^2_{\tilde{\delta}}(M_{B,n}),$$

but this embedding is not uniform in $\tilde{\delta}$. This change of weight does not interfere with the injectivity proof, but complicates the surjectivity case. We will need regularity results to regain the loss of weight.

Just as in the Sobolev case, we extend Theorems 4.18, 4.25 and 4.26 to $M_{B,n}$. Again we consider $M_{B,n}$ as the union of a compact set with the asymptotic region $P'_\infty/\mathbb{Z}_2$ and combine these results with the standard elliptic estimates on compact spaces. Once more, we give these results without proof:

**Lemma 5.11.** *Let $\epsilon \in (0,1)$, $k \in \mathbb{N}_{\geq 2}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$. There exists a constant $C_\epsilon > 0$ such that for any bounded $u \in C^{k,\alpha}_{loc}(M_{B,n})$ with $\Omega^{-2}\Delta^g u \in C^{k-2,\alpha}_\delta(M)$,*

$$u \in C^{k,\alpha}_\delta(M_{B,n})$$

*and*

$$\|u\|_{C^{k,\alpha}_\delta(M_{B,n})} \leq C_\epsilon \left[ \|\Omega^{-2}\Delta^g u\|_{C^{k-2,\alpha}_\delta(M_{B,n})} + \|u\|_{C^0_\delta(M_{B,n})} \right].$$

**Lemma 5.12.** *Let $\epsilon \in (0,1)$. (If $B = \mathbb{R} \times T^2$, also assume that $\epsilon$ is sufficiently small.) Fix $\delta \in \mathbb{R}\backslash\mathbb{Z}$ and $\alpha \in (0,1)$. There exist a constant $C_\epsilon > 0$ and a compact set $K$ such that for any $u \in C^{2,\alpha}_\delta(M_{B,n})$,*

$$\|u\|_{C^{2,\alpha}_\delta(M_{B,n})} \leq C_\epsilon \left[ \|\Omega^{-2}\Delta^g u\|_{C^{0,\alpha}_\delta(M_{B,n})} + \|u\|_{C^0_\delta(K)} \right].$$

The injectivity of $\Omega^{-2}\Delta^g$ follows immediately from the embedding of Hölder into Sobolev spaces:

**Theorem 5.13.** *Assume that $\delta < 0$, $k \in \mathbb{N}_{\geq 2}$ and $\alpha \in (0,1)$ When $B = \mathbb{R}^3$ the operator*

$$\Omega^{-2}\Delta\colon C^{k,\alpha}(M_{B,n}) \to C^{k-2,\alpha}(M_{B,n})$$

*has a trivial kernel. When $B \neq \mathbb{R}^3$ the operator*

$$\Omega^{-2}\Delta\colon C^{k,\alpha}(M_{B,n}) \oplus \mathbb{R}\phi \to C^{k-2,\alpha}(M_{B,n})$$

*has a trivial kernel.*

*Proof.* Let $u$ be a harmonic function in the domain of $\Omega^{-2}\Delta^g$. From the embedding $C^0_\delta(M_{B,n}) \subset L^2_{\delta/2}(M_{B,n})$, $u$ must be a harmonic function inside $W^{2,2}_{\delta/2}(M_{B,n})$ or $W^{2,2}_{\delta/2}(M_{B,n}) \oplus \mathbb{R}\phi$ respectively. By Theorems 5.7 and 5.10, $u$ must vanish. □

By embedding Hölder into Sobolev spaces we can always find an inverse, but this inverse might have the wrong decay rate. In order to use Lemmas 5.11 and 5.12 to regain the correct weight, we need to ask how the constant $C_\epsilon$ behaves under variation of $\delta$. We claim that $C_\epsilon$ can be picked uniformly in $\delta$.

**Proposition 5.14.** *Pick $\epsilon$, $k$, $\alpha$ and $\delta$ as described in Lemmas 5.11 or 5.12. Moreover, assume that $\delta$ is restricted to some closed interval. Then the constant $C_\epsilon$ in Lemmas 5.11 and 5.12 can be chosen uniformly with respect to $\delta$.*

*Proof.* We focus on Lemma 5.11, as the other will follow similarly. By the relationship between weighted norms and weighted operators, the estimate in Lemma 5.11 is equivalent to

$$\|u\|_{C^{k,\alpha}_{cf}(M_{B,n})} \leq C_\epsilon \left[\|L_\delta u\|_{C^{k-2,\alpha}_{cf}(M_{B,n})} + \|u\|_{C^0_{cf}(M_{B,n})}\right].$$

By Proposition 5.3, the operator $L_\delta$ is strictly elliptic, uniformly in $\delta$. This implies that the local regularity estimates, from which Lemmas 5.11 and 5.12 originate, are uniform in $\delta$. Therefore, Lemmas 5.11 and 5.12 must be uniform in $\delta$.

Alternatively, assume that $C_\epsilon$ is not uniform in $\delta$. Then, there must be sequences

$u_i \in C^{k,\alpha}_{cf}(M_{B,n})$ and $\delta_i \in \mathbb{R}$ such that

$$\|u_i\|_{C^{k,\alpha}_{cf}(M_{B,n})} = 1, \qquad \|L_{\delta_i} u_i\|_{C^{k-2,\alpha}_{cf}(M_{B,n})} \to 0,$$
$$\|u_i\|_{C^{0}_{cf}(M_{B,n})} \to 0, \qquad \delta_i \to \delta.$$

We apply Lemma 5.11 on $u_i$ with the limiting weight $\delta$, which yields

$$\begin{aligned}\|u_i\|_{C^{k,\alpha}_{cf}(M_{B,n})} &\leq C_\epsilon(\delta)\left[\|L_\delta u_i\|_{C^{k-2,\alpha}_{cf}(M_{B,n})} + \|u_i\|_{C^0_{cf}(M_{B,n})}\right]\\ &\leq C_\epsilon(\delta)\left[\|L_{\delta_i} u_i\|_{C^{k-2,\alpha}_{cf}(M_{B,n})} + \|(L_\delta - L_{\delta_i})u_i\|_{C^{k-2,\alpha}_{cf}(M_{B,n})} + \|u_i\|_{C^0_{cf}(M_{B,n})}\right].\end{aligned}$$

In the proof of proposition 5.3, there is an comparison between $L_\delta$ and the unweighted operator $\Omega^{-2}\Delta^g$. Using this explicit description of $L_\delta$ one can find a bounded, first-order differential operator $B$ such that

$$L_{\delta_i} = L_\delta + (\delta - \delta_i)\cdot B,$$

and the norm of $B$ only depends on the ellipticity of the unweighted operator $\Omega^{-2}\Delta^g$ and the interval on which $\delta_i$ resides. Therefore, $\|(L_\delta - L_{\delta_i})u_i\|_{C^{k-2,\alpha}_{cf}(M_{B,n})}$ converges to zero. By assumption, $\|L_{\delta_i}u_i\|_{C^{k-2,\alpha}_{cf}(M_{B,n})}$ and $\|u_i\|_{C^0_{cf}(M_{B,n})}$ also converge to zero and hence our regularity estimate implies $u_i$ converges to zero. This is a contradiction, as the norm of $u_i$ is fixed to one. □

With the uniform control of $C_\epsilon$ we can finally prove the bijectivity of $\Omega^{-2}\Delta^g$.

**Theorem 5.15.** *Let $\delta \in (-1, 0)$ (with $|\delta|$ sufficiently small if $B \neq \mathbb{R}^3$), $k \in \mathbb{N}_{\geq 2}$ and $\alpha \in (0,1)$. For any $f \in C^{k-2,\alpha}_\delta(M_{B,n})$, there exists an unique $u \in C^{k,\alpha}_\delta(M_{B,n})$ (or $u \in C^{k,\alpha}_\delta(M_{B,n}) \oplus \mathbb{R}\phi$ when $B \neq \mathbb{R}^3$) such that*

$$\Omega^{-2}\Delta^g u = f.$$

*Proof.* Let $f \in C^{k-2,\alpha}_\delta(M_{B,n})$. For any $\tilde\delta \in (\delta, \delta/2)$, $f$ is an element of $L^2_{\tilde\delta}(M_{B,n})$, and according to Theorems 5.7 and 5.9 there exists a $u \in W^{2,2}_{\tilde\delta}(M_{B,n})$ such that

$$\Omega^{-2}\Delta^g u = f.$$

By Lemma 5.5, $u$ is an element of $C^{2,\alpha}_{\tilde\delta}(M_{B,n})$. Because $\Omega^{-2}\Delta$ is injective, the function $u$ does not depend on the choice of $\tilde\delta$.

We claim $u \in C^0_\delta(M_{B,n})$. Indeed, let $x \in M_{B,n}$ and consider $|e^{-\delta\rho}u|(x)$. We can estimate this as

$$|e^{-\delta\rho}u|(x) \leq |e^{(\tilde{\delta}-\delta)\rho}|(x) \cdot \|e^{-\tilde{\delta}}u\|_{C^{2,\alpha}_{cf}(M_{B,n})}.$$

Because $\tilde{\delta}$ is not an indicial root, we can apply Lemma 5.12:

$$\begin{aligned} |e^{-\delta\rho}u|(x) &\leq |e^{(\tilde{\delta}-\delta)\rho}|(x) \cdot C_\epsilon \left[\|e^{-\tilde{\delta}\rho} f\|_{C^{2,\alpha}_{cf}(M_{B,n})} + \|e^{-\tilde{\delta}\rho}u\|_{C^0(K)}\right] \\ &\leq |e^{(\tilde{\delta}-\delta)\rho}|(x) \cdot C_\epsilon \left[\|e^{(\delta-\tilde{\delta})\rho}\|_{C^{2,\alpha}_{cf}(M_{B,n})} \cdot \|f\|_{C^{2,\alpha}_{\delta}(M_{B,n})} + \|e^{-\tilde{\delta}\rho}\|_{C^0(K)} \cdot \|u\|_{C^0(K)}\right]. \end{aligned}$$

On any fixed compact set, $e^{-\tilde{\delta}\rho}$ is bounded uniformly in $\tilde{\delta}$, because $e^{-\tilde{\delta}\rho}$ is continuous in $\delta$. The term $\|e^{(\delta-\tilde{\delta})\rho}\|_{C^{2,\alpha}_{cf}(M_{B,n})}$ is bounded uniformly in $\tilde{\delta}$, because $e^{(\delta-\tilde{\delta})\rho}$ decays when $\tilde{\delta} > \delta$. Also, the constant $C_\epsilon$ can be chosen uniformly with respect to $\tilde{\delta}$ due to Proposition 5.14. Therefore, there exists a constant $C(\epsilon, f, u|_K)$ that depends on $C_\epsilon$, $\|f\|_{C^{2,\alpha}_\delta(M_{B,n})}$ and $\|u\|_{C^0(K)}$ such that

$$|e^{-\delta\rho}u|(x) \leq |e^{(\tilde{\delta}-\delta)\rho}|(x) \cdot C(\epsilon, f, u|_k).$$

For each $x \in M_{B,n}$, we pick $\tilde{\delta} > \delta$ such that $|e^{(\tilde{\delta}-\delta)\rho}|(x) \leq 2$. This gives us an estimate of $|e^{-\delta\rho}u|(x)$ which is uniform in $\tilde{\delta}$ and so

$$\|u\|_{C^0_\delta(M_{B,n})} = \sup_{x \in M_{B,n}} |e^{-\delta\rho}u|(x) \leq 2C(\epsilon, f, u|_k) < \infty.$$

This proves the claim $u \in C^0_\delta(M_{B,n})$. From the regularity estimate in Lemma 5.11, $u \in C^{k,\alpha}_\delta$, which shows surjectivity. Injectivity is shown in Theorem 5.13. □

## 5.4 Bounded inverse estimate

Now we have shown that $\Omega^{-2}\Delta^g$ is an isomorphism, we can study the norm of its inverse. By the open mapping theorem, $\Omega^{-2}\Delta^g$ has a bounded inverse for every $\epsilon \in (0, \epsilon_0)$. To make the inverse function theorem work, we need the inverse to be bounded *uniformly* in the collapsing parameter $\epsilon$. We dedicate this section to the proof of this:

**Theorem 5.27.** *Let $\delta \in (-1, 0)$ (with $|\delta|$ sufficiently small if $B \neq \mathbb{R}^3$), $k \in \mathbb{N}_{\geq 2}$, and $\alpha \in (0, 1)$. There exist $\epsilon_0, C > 0$ such that for any collapsing parameter*

*$\epsilon \in (0, \epsilon_0)$ and $u \in C^{k,\alpha}_\delta(M_{B,n})$ (or $u + \lambda\phi \in C^{k,\alpha}_\delta(M_{B,n}) \oplus \mathbb{R}\phi$ when $B \neq \mathbb{R}^3$),*

$$\begin{aligned} \|u\|_{C^{k,\alpha}_\delta(M_{B,n})} &\leq C \left\|\Omega^{-2}\Delta^g u\right\|_{C^{k-2,\alpha}_\delta} && \text{if } B = \mathbb{R}^3 \\ \|u\|_{C^{k,\alpha}_\delta(M_{B,n})} + |\lambda| &\leq C \left\|\Omega^{-2}\Delta^g (u + \lambda\phi)\right\|_{C^{k-2,\alpha}_\delta} && \text{otherwise .} \end{aligned}$$

The proof of this theorem can be split into the following steps.

1. Assume that there is no uniform bounded inverse. There must be a sequence of functions $u_i$ and a sequence $\epsilon_i > 0$, such that $u_i$ has norm one, but $\Delta u_i$ and $\epsilon_i$ converge to zero[22].

2. Using the regularity estimate, construct a sequence of points $x_i$ at which the functions $|u_i|$ are uniformly bounded below, away from zero.

3. Modify the functions $u_i$, such that their domain is on a fixed limiting space.

4. Use the Arzela–Ascoli theorem to find a subsequence that converges to a non-zero harmonic function $u$.

5. Argue that the limiting space has no non-zero harmonic functions, and reach a contradiction.

Depending on whether the $x_i$ will concentrate near one of the singularities, we will pick different limiting spaces and apply different transformations to $u_i$. But for each case, we will follow the above steps.

*Remark* 5.16. The proof for the case $B = \mathbb{R}^3$ will be a simplified version of the proof for the case $B \neq \mathbb{R}^3$. Hence the rest of this section we only consider the latter case.

**Step 1.**

[22] When $\epsilon$ does not tend to zero, the operator $\Omega^{-2}\Delta^g$ is continuous in $\epsilon$ and hence, the existence of the uniform bounded inverse can be done by taking limits. We only need to consider the non-trivial case, when the collapsing parameter $\epsilon$ tends to zero.

**Lemma 5.17.** *Suppose that Theorem 5.27 is false. Then there exists sequences $u_i \in C^{k,\alpha}_\delta(M_{B,n})$, $\lambda_i \in \mathbb{R}$, $\epsilon_i \in (0,1)$ and $c > 0$ such that*

$$\begin{aligned} \|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})} + |\lambda_i| &= 1, \\ \|\Omega^{-2}\Delta^g(u_i + \lambda_i\phi)\|_{C^{k-2,\alpha}_\delta(M_{B,n})} &\to 0, \\ \|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})} &> c, \text{ and} \\ \epsilon_i &\to 0. \end{aligned}$$

*Proof.* The first two conditions follow directly from the negation of Theorem 5.27. We only need to show $\|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})} > c$. Suppose not, and assume that $\|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})}$ converges to zero. Because $|\lambda_i| \leq 1$, there must be a converging subsequence with limit $\lambda$. Because $\|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})} + |\lambda_i| = 1$ and $\|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})}$ converges to zero, the limit $\lambda$ must be equal to $\pm 1$.

We claim that this implies $\Delta^g\phi = 0$. Because $\phi$ is supported on the asymptotic region of $M_{B,n}$, we can estimate

$$\begin{aligned} \|\Omega^{-2}\Delta^g\phi\|_{C^0_\delta(B)} =& \|\Omega^{-2}\Delta^g(\lambda\phi)\|_{C^0_\delta(M_{B,n})} \\ \leq& |\lambda - \lambda_i| \cdot \|\Omega^{-2}\Delta^g\phi\|_{C^0_\delta(M_{B,n})} + \|\Omega^{-2}\Delta^g(u_i + \lambda_i\phi)\|_{C^{k-2,\alpha}_\delta(M_{B,n})} \\ &+ \|\Omega^{-2}\Delta^g\|_{op} \cdot \|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})}. \end{aligned}$$

The right hand side of this inequality converges to zero, and therefore $\Delta^g\phi = 0$. The function $\phi$ is not a harmonic function, which yields a contradiction. Hence, $\|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})}$ is uniformly bounded away from zero. □

**Step 2.** Next we study the property $\|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})} > c$ in more detail. Recall that the $C^{k,\alpha}_\delta$ is defined in terms of a supremum, i.e.

$$\|u\|_{C^{k,\alpha}_{cf}(U)} := \sum_{j=0}^{k} \sup_{x\in U} \|\nabla^j u(x)\|_{cf} + \sum_{\substack{x,y\in U \\ d(x,y)<\mathrm{InjRad}_x(g_{cf})}} \frac{\|\nabla^k u(x) - \nabla^k u(y)\|_{cf}}{d(x,y)^\alpha}.$$

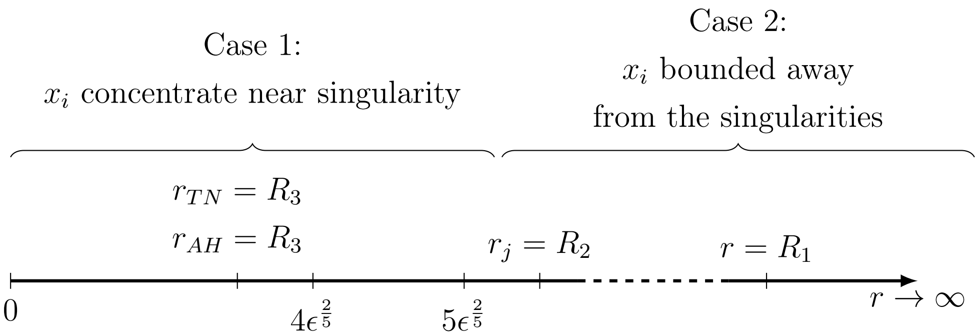
This is equivalent to the norm

$$\sup_{x\in U}\left[\sum_{j=0}^{k}\|\nabla^j u(x)\|_{cf}+\sum_{\substack{y\in U\\ d(x,y)<\mathrm{InjRad}_x(g_{cf})}}\frac{\|\nabla^k u(x)-\nabla^k u(y)\|_{cf}}{d(x,y)^\alpha}\right]$$

which enables us to define a 'pointwise norm':

$$\|u_i\|_{C^{k,\alpha}_{cf}(\{x\})}:=\sum_{j=0}^{k}\|\nabla^j u(x)\|_{cf}+\sum_{\substack{y\in U\\ d(x,y)<\mathrm{InjRad}_x(g_{cf})}}\frac{\|\nabla^k u(x)-\nabla^k u(y)\|_{cf}}{d(x,y)^\alpha}.$$

Similarly, we can define a weighted 'pointwise norm'. Using these 'norms', the condition $\|u_i\|_{C^{k,\alpha}_{\delta}(M_{B,n})} > c$ implies there is a sequence $x_i \in M_{B,n}$ such that $\|u_i\|_{C^{k,\alpha}_{\delta}(\{x_i\})} > \frac{c}{2} > 0$. The sequence of points $x_i$ can behave in two different ways:




1. The sequence $x_i$ **concentrates near a singularity**. That is, there is a subsequence of $x_i$ such that the radial coordinate $r_i$ or $r_j$ at $x_i$ converges to zero.
2. The sequence $x_i$ **is bounded away from the singularities**. That is, there is a subsequence of $x_i$ such that the radial coordinate at $x_i$ is uniformly bounded below.

At least one of these cases must happen, and we study them separately.

*Remark* 5.18. Normally, people also consider a third case when $x_i$ concentrate on the gluing region. We however view this as a special situation of case 1.

*Remark* 5.19. The case when $x_i$ concentrate near a non-fixed point singularity $p_i$, is similar to the case when $x_i$ concentrate near a fixed point singularity $q_j$. Therefore, we only explain the latter case.

**Case 1: $x_i$ concentrates near a singularity.**

**Step 3.** We consider the case when $x_i$ concentrates near a singularity. In this case $u_i|_{\{r_j \leq 2r_j(x_i)\}}$ is uniformly bounded away from zero in the $C^{k,\alpha}_\delta$ norm. At the same time, $\{r_j \leq 2r_j(x_i)\}$ can be viewed as a subset of the Atiyah-Hitchin manifold. Therefore, we use the Atiyah-Hitchin manifold as our limiting space.

To make our contradiction argument work, we need the norms, operators and weights on the limiting space to be invariant with respect to $\epsilon$. We constructed $g_{cf}$ such that this is true. We also chose $\Omega$ such that $\Omega^{-2}\Delta^g$ is $\epsilon$-invariant. However, the radial parameter $\rho$ does depend on $\epsilon$. To solve this we define a new $\epsilon$-invariant radial parameter $\rho_{AH} := \rho - \log\left(\frac{\epsilon}{1+\epsilon\alpha_j}\right)$ and we equip the Atiyah-Hitchin manifold with the weighted norm

$$\|u\|_{C^{k,\alpha}_\delta(AH)} = \|e^{-\delta\rho_{AH}}u\|_{C^{k,\alpha}_{g_{cf}}(AH)}.$$

Luckily, the weighted operator $L_\delta$ is the same whether we use $\rho$ or $\rho_{AH}$.

Next, we will restrict $u_i$ such that it is fully supported on the Atiyah-Hitchin manifold. For this we consider the family of smooth step functions $\chi_i$ on $M_{B,n}$ that are equal to 1 when $r_j \leq 2r_j(x_i)$ and equal to 0 when[23] $r_j \geq R_2$. Then $u_i \cdot \chi_i$ are compactly supported functions on the Atiyah-Hitchin manifold. Because $u_i \in C^{k,\alpha}_\delta(M_{B,n})$ is equivalent to $e^{-\delta\rho}u_i \in C^{k,\alpha}_{cf}(M_{B,n})$, we get

$$e^{-\delta\,\rho}u_i = e^{-\delta\,\rho_{AH}}\left(\frac{\epsilon}{1+\epsilon\alpha_j}\right)^{-\delta} u_i \in C^{k,\alpha}_{cf}(M_{B,n}).$$

With this insight, we consider a new sequence of functions $\tilde{u}_i := \chi_i \cdot \left(\frac{\epsilon}{1+\epsilon\alpha_j}\right)^{-\delta} u_i$ defined on the Atiyah-Hitchin manifold. In the following lemma, we show that $\tilde{u}_i$ has the same properties as $u_i$:

**Lemma 5.20.** *Suppose that Theorem 5.27 is false and that $x_i$ concentrate near a singularity $q_j$. Let $AH$ be the Atiyah-Hitchin manifold. Then the sequence*

[23] $R_2$ is defined in Definition 5.1.

$\tilde{u}_i := \chi_i \cdot \left(\frac{\epsilon}{1+\epsilon\alpha_j}\right)^{-\delta} u_i \in C^{k,\alpha}_\delta(AH)$ *satisfies*

$$\begin{aligned} \|\Omega^{-2}\Delta^g \tilde{u}_i\|_{C^{k-2,\alpha}_\delta(AH)} &\to 0, \\ \|\tilde{u}_i\|_{C^{k,\alpha}_\delta(AH)} &\leq 1, \ \textit{and} \\ \|\tilde{u}_i\|_{C^{k,\alpha}_\delta(AH)} &> \frac{c}{2}. \end{aligned}$$

*Proof.* We only modified $u_i$ outside of the region where $x_i$ concentrates and hence,

$$\|\tilde{u}_i\|_{C^{k,\alpha}_\delta(AH)} \geq \|\tilde{u}_i\|_{C^{k,\alpha}_\delta(\{r_j<2r_j(x_i)\})} > \frac{c}{2} > 0.$$

The step function $\chi_i$ is chosen such that $\mathrm{d}\,\chi_i$ and its derivatives are of order $(\log(R_2)-\log(2r_j(x_i)))^{-1}$ with respect to $g_{cf}$ and this converges to zero. Hence,

$$\|\tilde{u}_i\|_{C^{k,\alpha}_\delta(AH)} = \|\chi_i{\cdot}u_i\|_{C^{k,\alpha}_\delta(M_{B,n})} \leq \|\chi_i\|_{C^{k,\alpha}_{cf}(M_{B,n})}{\cdot}\|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})} = \|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})} = 1.$$

To estimate $\|\Omega^{-2}\Delta^g\tilde{u}_i\|_{C^{k-2,\alpha}_\delta(AH)}$, notice that on the support of $\chi_i$ the function $\phi$ is identically zero and

$$\Omega^{-2}\Delta^g(\chi_i \cdot u_i) = \chi_i \cdot \Omega^{-2}\Delta^g(u_i) + u_i \cdot \Omega^{-2}\Delta^g(\chi_i) - 2\Omega^{-2}\langle \mathrm{d}\,\chi_i, \mathrm{d}\,u_i\rangle_g.$$

Using that $g$ and $g^{q_j}$ are equivalent norms and that $\mathrm{d}\,\chi_i$ is decaying, we estimate

$$\begin{aligned} \|\Omega^{-2}\Delta^g(\tilde{u}_i)\|_{C^{k,\alpha}_\delta(AH)} \leq& \|\chi_i\|_{C^{k,\alpha}_{cf}(M_{B,n})} \cdot \|\Omega^{-2}\Delta^g(u_i)\|_{C^{k,\alpha}_\delta(M_{B,n})} \\ &+ \|u_i\|_{C^{k,\alpha}_\delta(M_{B,n})} \cdot \|\Omega^{-2}\Delta^g(\chi_i)\|_{C^{k,\alpha}_{cf}(M_{B,n})} + \mathcal{O}\left(\frac{1}{-\log(r_j(x_i))}\right). \end{aligned}$$

By Proposition 5.3, $\Omega^{-2}\Delta^g\chi_i$ is uniformly bounded by $\|\mathrm{d}\,\chi_i\|_{C^{k,\alpha}_{cf}(M_{B,n})}$ and so

$$\|\Omega^{-2}\Delta^g\tilde{u}_i\|_{C^{k-2,\alpha}_\delta(AH)} \to 0.$$

□

**Step 4.** Next, we will find a subsequence of $\tilde{u}_i$ which converges to some $\tilde{u} \in C^{k,\alpha}_\delta(AH)$. We equipped the asymptotic region of the Atiyah-Hitchin metric with the Taub-NUT metric with negative mass, for which we developed a rich regularity theory in Chapter 4. This induces regularity estimates for $\Omega^{-2}\Delta^g$. For example, Theorem 4.25 induces the lemma:

**Lemma 5.21.** *There exists a uniform $C > 0$ and a compact set $K \subset AH$ such that for any $u \in C^{k,\alpha}_\delta(AH)$,*

$$\|u\|_{C^{k,\alpha}_\delta(AH)} \leq C\left[\|\Omega^{-2}\Delta^g u\|_{C^{k-2,\alpha}_\delta(AH)} + \|u\|_{C^0_\delta(K)}\right].$$

*Proof.* Because we equipped the asymptotic region the Atiyah-Hitchin metric with the Taub-NUT metric with negative mass, we can apply Theorem 4.25 to get

$$\|u\|_{C^{k,\alpha}_\delta(AH\setminus B_R(0))} \leq C\left[\|\Omega^{-2}\Delta^{g^{q_j}} u\|_{C^{k-2,\alpha}_\delta(AH\setminus B_{R'}(0))} + \|u\|_{C^0_\delta(K)}\right]$$

for sufficiently large $0 < R' < R$. At the same time $\Omega^{-2}\Delta^g$ is strictly elliptic, and the metric $g_{cf}$ is independent of $\epsilon$ on $B_R(0)$. This implies that the Schauder estimate

$$\|u\|_{C^{k,\alpha}_\delta(B_R(0))} \leq C\left[\|\Omega^{-2}\Delta^g u\|_{C^{k-2,\alpha}_\delta(B_{2R}(0))} + \|u\|_{C^0_\delta(B_{2R}(0))}\right]$$

is uniform in $\epsilon$. Combining our results yields,

$$\begin{aligned}\|u\|_{C^{k,\alpha}_\delta(AH)} \leq& \|u\|_{C^{k,\alpha}_\delta(B_R(0))} + \|u\|_{C^{k,\alpha}_\delta(AH\setminus B_R(0))}\\ \leq& C\left[\|\Omega^{-2}\Delta^g u\|_{C^{k-2,\alpha}_\delta(AH)} + \|u\|_{C^0_\delta(K)}\right].\end{aligned}$$

□

Using the Arzela-Ascoli theorem, there exists a subsequence of $\tilde{u}_i$ which converges to some $\tilde{u} \in C^0_\delta(K)$ for any compact set $K$. When we apply Lemma 5.21 on this sequence, we get

$$\|\tilde{u}_i - \tilde{u}_j\|_{C^{k,\alpha}_\delta(AH)} \leq C\left[\|\Omega^{-2}\Delta^g(\tilde{u}_i - \tilde{u}_j)\|_{C^{k-2,\alpha}_\delta(AH)} + \|\tilde{u}_i - \tilde{u}_j\|_{C^0_\delta(K)}\right] \to 0.$$

This implies that $\tilde{u}_i$ is a Cauchy sequence in $C^{k,\alpha}_\delta(AH)$ and hence it converges to some $\tilde{u} \in C^{k,\alpha}_\delta(AH)$.

**Step 5.** This limiting function is harmonic, because $\Omega^{-2}\Delta$ is a continuous operator. By assumption, $\delta < 0$, and hence $\tilde{u}$ must be decaying. By the maximum principle $\tilde{u}$ must vanish everywhere. We conclude $\|\tilde{u}_i\|_{C^{k,\alpha}_\delta(AH)}$ converges to zero, which contradicts the fact that $\|\tilde{u}_i\|_{C^{k,\alpha}_\delta(AH)} > c/2 > 0$. Therefore, the sequence $x_i$ cannot concentrate near the singularities.

**Case 2: $x_i$ is bounded away from the singularities**

**Step 3.** Next we consider the case in which $x_i$ is bounded away from the singularities. Again we need to modify $u_i$, such that their domain is defined on a fixed limiting space. The points $\{x_i\}$ lie inside the circle bundle $P$, on which the Gibbons-Hawking metric is defined. The radius of the fibers of $P$ are $\mathcal{O}(\epsilon)$, and hence we expect that, in the limit $\epsilon_i \to 0$, $P$ collapses to its base space $B'$. At the same time, the neighbourhoods we removed from $B$ to get $B'$ shrink at rate $\mathcal{O}(\epsilon^{2/5})$, and hence we pick the flat space $B$ as our our limiting space.

Next we construct a step function such that we can view $u_i$ as functions on $P$. Because the points $\{x_i\}$ are bounded away from the singularities, there is a constant $R_B$ such that $r_j(x_i) > R_B$. Therefore, consider the family of smooth step functions $\chi_i$ on $M_{B,n}$ that are equal to one when $r_i, r_j \geq R_B$ and equal to zero when $r_i, r_j \leq 5\epsilon^{2/5}$. We consider a new sequence of functions $\tilde{u}_i := u_i \cdot \chi_i$ on $P$. Also, let $\tilde{u}_i^b$ be the $S^1$-invariant[24] part of $\tilde{u}_i$. In the following lemma, we explain the behaviour of $\tilde{u}_i$ and $\tilde{u}_i^b$:

**Lemma 5.22.** *Suppose that Theorem 5.27 is false and $x_i$ are bounded away from the singularities. Then, the sequences $\tilde{u}_i := \chi \cdot u_i \in C^{k,\alpha}_\delta(M_{B,n})$ and its $S^1$ invariant part $\tilde{u}_i^b \in C^{k,\alpha}_\delta(B')$ satisfy*

$$\|\Omega^{-2}\Delta^g(\tilde{u}_i + \lambda_i\phi)\|_{C^{k-2,\alpha}_\delta(P)} \to 0, \qquad \|\Omega^{-2}\Delta^g(\tilde{u}_i^b + \lambda_i\phi)\|_{C^{0,\alpha}_\delta(B')} \to 0,$$
$$\frac{c}{2} \leq \|\tilde{u}_i\|_{C^{k,\alpha}_\delta(P)} \leq 1, \text{ and} \qquad \|\tilde{u}_i - \tilde{u}_i^b\|_{C^0_\delta(P)} \to 0.$$

*Proof.* The arguments that $\|\Omega^{-2}\Delta^g(\tilde{u}_i+\lambda_i\phi)\|_{C^{k-2,\alpha}_\delta(P)} \to 0$ and $\frac{c}{2} \leq \|\tilde{u}_i\|_{C^{k,\alpha}_\delta(P)} \leq 1$ are identical to the arguments given in Lemma 5.20. For the estimate on $\|\Omega^{-2}\Delta^g(\tilde{u}_i^b+\lambda_i\phi)\|_{C^{0,\alpha}_\delta(B)}$, recall that $g_{cf}$ is constructed from $S^1$-invariant metrics. Therefore, the projection operator $\pi_b$, that is defined in Definition 4.21, commutes with the Lapla-

[24] See definition 4.21.

cian. By Lemma 4.23, the operator $\pi_b$ is uniformly bounded and hence

$$\begin{aligned}\|\Omega^{-2}\Delta^g(\tilde{u}_i^b+\lambda_i\phi)\|_{C^{0,\alpha}_\delta(B')} &\leq \|\pi_b(\Omega^{-2}\Delta^g(\tilde{u}_i+\lambda_i\phi))\|_{C^{0,\alpha}_\delta(M_{B,n})}\\ &\leq C\|\Omega^{-2}\Delta^g(\tilde{u}_i+\lambda_i\phi))\|_{C^{0,\alpha}_\delta(M_{B,n})}\to 0.\end{aligned}$$

Before we estimate $\tilde{u}_i^f := \tilde{u}_i - \tilde{u}_i^b$ we prove the claim that for every $x\in P$, there is a $t\in[0,2\pi]$ such that $\tilde{u}_i^f(e^{it}\cdot x)=0$. If not we can assume w.l.o.g. there is an $x\in P$ such that for all $t\in[0,\pi]$, the function $\tilde{u}_i^f(e^{it}\cdot x)>0$. By Definition 4.21,

$$\pi_b(\tilde{u}_i^f)(x)=\frac{1}{2\pi}\int_{t=0}^{2\pi}\tilde{u}_i(e^{it}\cdot x)\,\mathrm{d}\,t,$$

which is strictly greater than zero. At the same time, $\pi_b$ is a projection operator and so $\pi_b(\tilde{u}_i^f)=\pi_b(\tilde{u}_i-\pi_b(\tilde{u}_i))=0$. This is a contradiction and proves our claim.

Finally, we estimate $\tilde{u}_i-\tilde{u}_i^b$. Fix $x\in P$ and let $t_0\in[0,2\pi]$ such that $\tilde{u}_i(e^{it_0}\cdot x)=\tilde{u}_i^b(e^{it_0}\cdot x)$. By the fundamental theorem of calculus,

$$u_i(x)-u_i(e^{it_0}\cdot x)=-\int_{t=0}^{t_0}\frac{\partial}{\partial t}u_i(e^{it}\cdot x)\,\mathrm{d}\,t.$$

Using that $u_i^b$ is $S^1$-invariant,

$$|\tilde{u}_i(x)-\tilde{u}_i^b(x)|\leq\int_{t=0}^{2\pi}\mathrm{d}\,u_i(\partial_t)\,\mathrm{d}\,t\leq\|\mathrm{d}\,\tilde{u}_i\|_{g_{cf}}\cdot\int_{t=0}^{2\pi}\sqrt{g_{cf}(\partial_t,\partial_t)}\,\mathrm{d}\,t,$$

and we see that this integral is the length of the fiber at $x$. Because $g_{cf}$ is equivalent to $g_{cf}^{GH}$, the length of the fiber decays with order $\mathcal{O}(\epsilon)$. Hence,

$$\|\tilde{u}_i-\tilde{u}_i^b\|_{C^0_\delta(P)}\leq\|\tilde{u}_i\|_{C^{1,\alpha}_\delta(P)}\cdot\mathcal{O}(\epsilon),$$

which converges to zero. □

**Step 4.** Next we want to find a subsequence of $\tilde{u}_i^b+\lambda_i\phi$ which converges to some twice differentiable harmonic function on $B$. First we need to determine what the limiting metric will be. For this, notice that on the support of $\tilde{u}_i^b$ the metric $g_{cf}$ is an interpolation of metrics that can be decomposed into some uniform metric $\tilde{g}_B$ on the base space and a part that is of order $\epsilon$. For example, the metric

$$g_{cf}^{q_j} = r_j^{-2}\,\mathrm{d}\,r_j + g_{S^2} + \frac{\epsilon^2}{r_j h_\epsilon^2}\eta_j^2$$

can be written in the form $\tilde{g}_B + \mathcal{O}(\epsilon)$ and the limiting metric is $\tilde{g}_B = r_j^{-2}\,\mathrm{d}\,r_j + g_{S^2}$. We conclude that in the limit $\epsilon \to 0$, the metric $g_{cf}$ degenerates to a metric on $B \setminus \cup\{p_i, q_j\}$. Therefore, for any compact sets $K \subset K' \subset B \setminus \cup\{p_i, q_j\}$, we have the Schauder estimate

$$\|\tilde{u}_i^b - \tilde{u}_j^b\|_{C^{2,\alpha}_{\tilde{g}_B}(K)} \leq C\left[\|\Omega^{-2}\Delta^g(\tilde{u}_i^b - \tilde{u}_j^b)\|_{C^{0,\alpha}_{\tilde{g}_B}(K')} + \|\tilde{u}_i^b - \tilde{u}_j^b\|_{C^0_{\tilde{g}_B}(K')}\right].$$

By introducing $\lambda_i\phi$, we can rewrite this as

$$\begin{aligned}\|\tilde{u}_i^b - \tilde{u}_j^b\|_{C^{2,\alpha}_{\tilde{g}_B}(K)} \leq & C\Big[\|\Omega^{-2}\Delta^g(\tilde{u}_i^b + \lambda_i\phi)\|_{C^{0,\alpha}_{\tilde{g}_B}(K')} + \|\Omega^{-2}\Delta^g(\tilde{u}_j^b + \lambda_j\phi)\|_{C^{0,\alpha}_{\tilde{g}_B}(K')} \\ & + |\lambda_i - \lambda_j| \cdot \|\Omega^{-2}\Delta^g\phi\|_{C^{0,\alpha}_{\tilde{g}_B}(K')} + \|\tilde{u}_i^b - \tilde{u}_j^b\|_{C^0_{\tilde{g}_B}(K')}\Big].\end{aligned}$$

Combining this result with Arzela-Ascoli, there is a subsequence of $\tilde{u}_i^b$ which converges in $C^{2,\alpha}_{\tilde{g}_B}(K)$. By exhausting the punctured base space by compact sets, applying Arzela-Ascoli on each of them, and taking the diagonal sequence, we conclude:

**Lemma 5.23.** *There exists a twice differentiable function $\tilde{u}^b$ on $B \setminus \cup\{p_i, q_j\}$ and a $\lambda \in [-1, 1]$, such that for any compact set $K \subset B \setminus \cup\{p_i, q_j\}$,*

$$\begin{aligned}\tilde{u}_i^b \to & \tilde{u}^b \in C^{2,\alpha}_{\tilde{g}_B}(K) \\ \lambda_i \to & \lambda\end{aligned}$$

*and*

$$\Delta^B(\tilde{u}^b + \lambda\phi) = \Delta^g(\tilde{u}^b + \lambda\phi) = 0.$$

In the last part of the lemma, one has to notice that on the support of $u$ the metric $g$ is $g^{GH}$ and $\Delta^{GH} = \frac{1}{h_\epsilon}\Delta^B$ for $S^1$-invariant functions.

Before we can make any qualitative statement about $\tilde{u}^b + \lambda\phi$, we need to consider its behaviour near the boundary of $B \setminus \cup\{p_i, q_j\}$. The decay behaviour near infinity follows from Theorems 4.25 and 4.26 as their estimates can be rewritten as

$$\begin{aligned}\|\tilde{u}_i^b - \tilde{u}_j^b\|_{C^{2,\alpha}_\delta(P_\infty/\mathbb{Z}_2)} \leq & C\Big[\|\Omega^{-2}\Delta^g(\tilde{u}_i^b + \lambda_i\phi)\|_{C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + \|\Omega^{-2}\Delta^g(\tilde{u}_j^b + \lambda_j\phi)\|_{C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} \\ & + |\lambda_i - \lambda_j| \cdot \|\Omega^{-2}\Delta^g\phi\|_{C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + \|\tilde{u}_i^b - \tilde{u}_j^b\|_{C^0_\delta(K_\infty/\mathbb{Z}_2)}\Big].\end{aligned}$$

The right hand side of this equation converges to zero and hence $\tilde{u}_i^b$ is a Cauchy sequence in $C^{2,\alpha}_\delta(P_\infty/\mathbb{Z}_2)$. This implies $\tilde{u}^b$ decays with order $e^{\delta\rho}$. Next we study the behaviour near the punctures, where $g_{cf}$ is just the conformal metric $g_{cf}^{TN}$ or $g_{cf}^{TN'}$. We claim that $\tilde{u}_b$ can be smoothly extended over the singularities, and we show this in two steps: First we will show that $\tilde{u}^b$ has some polynomial divergence near a singularity. Secondly, we will show that this decay is slow enough for it to be a removable singularity. Again, the arguments are identical for fixed or non-fixed singularities. Therefore, we only explain one of these cases.

**Lemma 5.24.** *On any compact neighbourhood $K$ of $q_j$ inside $B$ and any $\tilde{\delta} < \delta$, $r_j^{-2\tilde{\delta}}\tilde{u}^b \in C^0(K)$.*

*Proof.* Let $\tilde{\epsilon} > 0$ be arbitrary. There is some small open ball $B(q_j)$, such that on this ball $|r_j^{\delta-\tilde{\delta}}| < \tilde{\epsilon}$. Hence, for any $k, l \in \mathbb{N}$,

$$\begin{aligned}\|r_j^{-\tilde{\delta}}(\tilde{u}_k^b - \tilde{u}_l^b)\|_{C^0(K)} \leq & \|r_j^{\delta-\tilde{\delta}}\|_{C^0(B(q_j))} \cdot \left(\|r_j^{-\delta}\tilde{u}_k^b\|_{C^0(B(q_j))} + \|r_j^{-\delta}\tilde{u}_l^b\|_{C^0(B(q_j))}\right) \\ & + \|r_j^{\delta-\tilde{\delta}}\|_{C^0(K\setminus B(q_j))} \cdot \|r_j^{-\delta}(\tilde{u}_k^b - \tilde{u}_l^b)\|_{C^0(K\setminus B(q_j))}.\end{aligned}$$

By Lemma 5.22, $r_j^{-\delta}\tilde{u}_k^b$ approximates $r_j^{-\delta}\tilde{u}_k$, which is uniformly bounded, i.e. there exists an $N \in \mathbb{N}$ such that for all $k > N$,

$$\|r_j^{-\delta}\tilde{u}_k^b\|_{C^0(B(q))} \leq \|\tilde{u}_k - \tilde{u}_k^b\|_{C^0_\delta(P)} + \|\tilde{u}_k\|_{C^0_\delta(P)} \leq 2.$$

Secondly, $K \setminus B(q_j)$ is compact, and according to Lemma 5.23, one can chose $N \in \mathbb{N}$ such that for all $k, l > N$,

$$\|r_j^{-\delta}(\tilde{u}_k^b - \tilde{u}_l^b)\|_{C^0(K\setminus B(q_j))} < \frac{\tilde{\epsilon}}{\|r_j^{\delta-\tilde{\delta}}\|_{C^0(K\setminus B(q_j))}}.$$

We conclude

$$\begin{aligned}\|r_j^{-\tilde{\delta}}(\tilde{u}_k^b - \tilde{u}_l^b)\|_{C^0(K)} \leq & \|r_j^{\delta-\tilde{\delta}}\|_{C^0(B(q_j))} \cdot \left(\|r_j^{-\delta}\tilde{u}_k^b\|_{C^0(B(q_j))} + \|r_j^{-\delta}\tilde{u}_l^b)\|_{C^0(B(q_j))}\right) \\ & + \|r_j^{\delta-\tilde{\delta}}\|_{C^0(K\setminus B(q_j))}\|r_j^{-\delta}(\tilde{u}_k^b - \tilde{u}_l^b)\|_{C^0(K\setminus B(q_j))} \\ \leq & 5\epsilon.\end{aligned}$$

Therefore, $r^{-\tilde{\delta}}\tilde{u}_k^b$ is a Cauchy sequence in $C^0(K)$ and its limit is $r_j^{-\tilde{\delta}}\tilde{u}^b$. □

**Lemma 5.25.** *The function $\tilde{u}^b$ can be smoothly extended to a $g_B$-harmonic function on $B$.*

*Proof.* Our goal is to show that $\tilde{u}^b$ is an element of $L^2_{g_{eucl}}$ on some compact set, containing a singularity $q_j$. If this is true, the standard Euclidean Schauder estimates will imply $\tilde{u}^b$ can be smoothly extended over $q_j$.

By Lemma 5.24, $r_j^{-\tilde{\delta}}\tilde{u}^b$ is bounded for all $\tilde{\delta} < \delta < 0$ and therefore the weighted norm

$$\|u\| := \int_{r_j=0}^{R_2}\int_{S^2} r_j^{-4\delta}(\tilde{u}^b)^2 \,\mathrm{d}\log r_j \wedge \mathrm{Vol}_{S^2}$$

is finite. Let $\psi_n$ be the spherical harmonics and expand $\tilde{u}^b$ in term of $\psi_n$, i.e.

$$\tilde{u}^b := \sum_{n=0}^{\infty}(a_n\, r_j^n + b_n\, r_j^{-n-1})\psi_n.$$

By the orthonormality of the spherical harmonics,

$$\begin{aligned}\|u\| &= \int_{r_j=0}^{R_2}\int_{S^2} r_j^{-4\delta}(\tilde{u}^b)^2 \,\mathrm{d}\log r_j \wedge \mathrm{Vol}_{S^2} \\ &= \sum_{n=0}^{\infty}\int_{r_j=0}^{R_2} r_j^{-4\delta}(a_n\, r_j^n + b_n\, r_j^{-n-1})^2 \,\mathrm{d}\log r_j.\end{aligned}$$

Because $|\delta| \ll 1$, this can only be finite if $b_n = 0$ for all $n \in \mathbb{N}_0$. This implies $\tilde{u}^b$ is in $L^2_{g_{eucl}}$ on any compact set containing the singularities. □

**Step 5.** With the asymptotic behaviour of $\tilde{u}^b$ understood, we can now prove that $\tilde{u}^b = \lambda\phi = 0$. Indeed, the function $\tilde{u}^b + \lambda\phi$ is of order $\mathcal{O}(e^{-\delta\rho})$ and the only harmonic functions of these kind are constant. Because the map $\phi$ is unbounded, $\lambda$ must be equal to zero. Finally, the function $\tilde{u}^b$ is decaying, and the only constant that is decaying is the constant zero function. Therefore, $\tilde{u}^b = \lambda\phi = 0$. The implication this has for $\tilde{u}_i$ will be summarised in the following lemma.

**Lemma 5.26.** *There exists a subsequences of $\tilde{u}_i$ and $\lambda_i$, such that for any compact set $K \subset P$,*

$$\tilde{u}_i \to 0 \in C^0_{\tilde{g}_B}(K), \text{ and } \lambda_i \to 0.$$

We finally prove that Lemmas 5.22 and 5.26 lead to a contradiction. Indeed, suppose that the sequence $x_i$ is unbounded. Then Lemma 5.22 implies

$$0 < \frac{c}{2} \leq \|\tilde{u}_i\|_{C^{k,\alpha}_\delta(P_\infty/\mathbb{Z}_2)}.$$

However, Theorems 4.25 and 4.26 imply

$$\|\tilde{u}_i\|_{C^{2,\alpha}_\delta(P_\infty/\mathbb{Z}_2)} \leq C \Big[\|\Omega^{-2}\Delta^{GH}(\tilde{u}_i + \lambda_i\phi)\|_{C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + |\lambda_i| \cdot \|\Omega^{-2}\Delta^{GH}\phi\|_{C^{0,\alpha}_\delta(P'_\infty/\mathbb{Z}_2)} + \|\tilde{u}_i\|_{C^0_\delta(K_\infty/\mathbb{Z}_2)}\Big],$$

which, according to Lemmas 5.22 and 5.26, converges to zero.

The only possibility left is when $x_i$ has a subsequence that converges to some point $x$ in the interior of $P$. Up to some local universal cover, the metric $g_{cf}$ on $P$ has bounded geometry and hence the Schauder estimates imply

$$\|\tilde{u}_i\|_{C^{2,\alpha}_\delta(B_r(x))} \leq C \Big[\|\Omega^{-2}\Delta^{GH}\tilde{(}u_i + \lambda_i\phi)\|_{C^{0,\alpha}_\delta(B_{2r}(x))} + |\lambda_i| \cdot \|\Omega^{-2}\Delta^{GH}\phi\|_{C^{0,\alpha}_\delta(B_{2r}(x))} + \|\tilde{u}_i\|_{C^0_\delta(B_{2r}(x))}\Big]$$

for some small radius $r > 0$. The right hand side of this inequality vanishes when $i$ tends to infinity and so $\|\tilde{u}_i\|_{C^{2,\alpha}_\delta(B_r(x))}$ will vanish also. This contradicts lemma 5.22, because for sufficiently large $i$ the points $x_i$ will concentrate inside $B_r(x)$, and hence

$$0 < \frac{c}{2} \leq \|\tilde{u}_i\|_{C^{k,\alpha}_\delta(B_r(x))}.$$

All possible cases lead to contradictions, and so we conclude:

**Theorem 5.27.** *Let $\delta \in (-1,0)$ (with $|\delta|$ sufficiently small if $B \neq \mathbb{R}^3$), $k \in \mathbb{N}_{\geq 2}$, and $\alpha \in (0,1)$. There exist $\epsilon_0, C > 0$ such that for any collapsing parameter $\epsilon \in (0,\epsilon_0)$ and $u \in C^{k,\alpha}_\delta(M_{B,n})$ (or $u + \lambda\phi \in C^{k,\alpha}_\delta(M_{B,n}) \oplus \mathbb{R}\phi$ when $B \neq \mathbb{R}^3$),*

$$\begin{aligned} \|u\|_{C^{k,\alpha}_\delta(M_{B,n})} &\leq C \, \|\Omega^{-2}\Delta^g u\|_{C^{k-2,\alpha}_\delta} && \text{if } B = \mathbb{R}^3 \\ \|u\|_{C^{k,\alpha}_\delta(M_{B,n})} + |\lambda| &\leq C \, \|\Omega^{-2}\Delta^g (u+\lambda\phi)\|_{C^{k-2,\alpha}_\delta} && \text{otherwise }. \end{aligned}$$

## 5.5 Proof of the main theorem

In this section we prove Theorem 1.1 by showing that our approximate solutions from Chapter 3 can be perturbed into genuine gravitational instantons. In Section 2.2, we described this perturbation problem and showed that the hyperkähler condition can be phrased as an elliptic equation. We claimed that this elliptic equation can be solved using the inverse function theorem. Before we show that these claims are indeed true, we first recall some facts and notation. Namely, in Section 2.2, we introduced the projection operator

$$\begin{aligned} \mathrm{Tf}\colon \mathrm{Mat}_{3\times 3}(\mathbb{R}) \otimes \Omega^4(M) &\to \mathrm{Sym}^2_0(\mathbb{R}^3) \otimes \Omega^4(M) \\ P \otimes \mu &\mapsto \left(\frac{1}{2}P + \frac{1}{2}P^* - \frac{1}{3}\operatorname{Tr}(P)\operatorname{Id}\right) \otimes \mu, \end{aligned} \tag{1}$$

and the wedge operator

$$\begin{aligned} \Lambda\colon \Omega^+(M) \otimes \mathbb{R}^3 &\to \mathrm{Mat}_{3\times 3}(\mathbb{R}) \otimes \Omega^4(M) \\ \sigma &\mapsto \sigma \wedge \omega, \end{aligned} \tag{ 4}$$

and our goal was to find a triple of self-dual 2-forms $\zeta$ such that

$$\frac{1}{2}\Lambda^{-1}\operatorname{Tf}(\omega\wedge\omega) + \Delta\zeta + 2\Lambda^{-1}\operatorname{Tf}(\mathrm{d}\,\mathrm{d}^*\zeta \wedge \mathrm{d}\,\mathrm{d}^*\zeta) = 0. \tag{6}$$

We showed that if $\zeta$ satisfied Equation 6, then $\omega + 2\,\mathrm{d}\,\mathrm{d}^*\zeta$ is a hyperkähler triple. In order to solve this equation, we use the following version of the inverse function theorem:

**Theorem 2.15** (Inverse function theorem). *Let $F(x) = F(0) + L(x) + N(x)$ be a smooth function between Banach spaces such that there exist $r, q, C > 0$ satisfying*

1. *$L$ is an invertible linear operator with $\|L^{-1}\| < C$,*
2. *$\|N(x) - N(y)\| \leq q \cdot \|x + y\| \cdot \|x - y\|$ for all $x, y \in B_r(0)$, and*
3. *$\|F(0)\| < \min\left\{\frac{1}{4qC^2}, \frac{r}{2C}\right\}$.*

*Then, there exists a unique $x$ in the domain of $F$ such that $F(x) = 0$ and $\|x\| \leq 2C\|F(0)\|$.*

Let

$$F(\zeta) := \Omega^{-2}\Delta\zeta + \frac{1}{2}\Omega^{-2}\Lambda^{-1}\operatorname{Tf}(\omega \wedge \omega) + 2\Omega^{-2}\Lambda^{-1}\operatorname{Tf}(\mathrm{d}\,\mathrm{d}^*\zeta \wedge \mathrm{d}\,\mathrm{d}^*\zeta),$$

and identify the constant, linear and non-linear parts as

$$\begin{aligned} F(0) =& \frac{1}{2}\Omega^{-2}\Lambda^{-1}\operatorname{Tf}(\omega \wedge \omega) \\ L(\zeta) =& \Omega^{-2}\Delta\zeta \\ N(\zeta) =& 2\Omega^{-2}\Lambda^{-1}\operatorname{Tf}(\mathrm{d}\,\mathrm{d}^*\zeta \wedge \mathrm{d}\,\mathrm{d}^*\zeta), \end{aligned}$$

where $\Delta$ is the Hodge Laplacian. According to Lemma 2.14, this Laplacian is equal to $\not{D}^2$ for some Dirac operator $\not{D}$. When we decompose $\zeta$ into $\sum_i u_i \cdot \omega_i$, where $u_i \in C^\infty(M_{B,n})$, the Weitzenböck formula yields

$$\not{D}^2 (u_i\, \omega_i) = (\Delta^g u_i)\, \omega_i - 2\nabla_{\nabla u_i}\omega_i,$$

because $\not{D}\, \omega_i = 0$. With Theorem 5.15 in mind we pick the domain of $F$ to be $C^{k+2,\alpha}_\delta(M_{B,n}) \cdot \omega \subseteq \Omega^+(M_{B,n}) \otimes \mathbb{R}^3$ (or $(C^{k+2,\alpha}_\delta(M_{B,n}) \oplus \mathbb{R}\phi) \cdot \omega$ when $B \neq \mathbb{R}^3$). That is, we pick self-dual 2-forms of the form $\sum_i u_i \cdot \omega_i$ and we define the norm of $\sum_i u_i \cdot \omega_i$ as $\sum_i \|u_i\|^2_{C^{k,\alpha}_\delta(M_{B,n})}$. Similarly, for the codomain we pick $(C^{k,\alpha}_\delta(M_{B,n}) \cdot \omega) \otimes \mathbb{R}^3$.

Having defined $F$ as a smooth map between Banach spaces, we show that the conditions of Theorem 2.15 are satisfied. Most of the work will be relating the various norms we used. We remind the reader that the calculations for all higher derivatives will follow from the calculations of the $C^0$ norm. Indeed, all the estimates used in this chapter are based from the error estimate in Theorem 3.23. As explained in that section, this error estimate originates from our estimates of the harmonic function $h$, for which all higher derivatives are known due to Remark 3.5. We chose our norms such that all derivatives of $h$ have the same decay rate and so the study of the $C^0$ norm is enough.

**The constant part**

First we need to estimate the constant term $\frac{1}{2}\Omega^{-2}\Lambda^{-1}\operatorname{Tf}(\omega\wedge\omega)$ in Equation 6 with respect to the norm $(C^{k,\alpha}_{\delta}\cdot\omega)\otimes\mathbb{R}^3$. To examine $\Lambda^{-1}\circ\operatorname{Tf}$ in local coordinates, let $\mu\in\Omega^4(M_{B,n})$ be a volume form and define $P\colon M_{B,n}\to\operatorname{Mat}_{3\times 3}(\mathbb{R})$ by

$$\omega_i\wedge\omega_j = P_{ij}\mu.$$

Then the composition of Tf and $\Lambda$ is given by

$$\begin{array}{ccccc} \operatorname{Mat}_{3\times3}(\mathbb{R})\otimes\Omega^4(M_{B,n}) & \xrightarrow{\operatorname{Tf}} & \operatorname{Sym}^2_0(\mathbb{R}^3)\otimes\Omega^4(M_{B,n}) & \xrightarrow{\Lambda^{-1}} & \Omega^+(M_{B,n})\otimes\mathbb{R}^3 \\ P\otimes\mu & \longmapsto & (P-\frac{1}{3}\operatorname{tr}(P)\operatorname{Id})\otimes\mu & \longmapsto & \sum_{ij}\left(\operatorname{Id}-\frac{1}{3}\operatorname{tr}(P)P^{-1}\right)_{ij}\omega_j\otimes e_i, \end{array}$$

where $e_i$ is the standard orthonormal basis on $\mathbb{R}^3$. In order to estimate $\frac{1}{2}\Omega^{-2}\Lambda^{-1}\operatorname{Tf}(\omega\wedge\omega)$, we need to estimate the components of $\operatorname{Id}-\frac{1}{3}\operatorname{tr}(P)P^{-1}$ with respect to the $C^{k,\alpha}_{\delta}(M_{B,n})$ norm. According to Theorem 3.23, $\omega$ is a hyperkähler triple outside the gluing regions, and so $\operatorname{Tf}(\omega\wedge\omega)=0$. Inside the gluing regions, the triple $\omega$ satisfies

$$\frac{1}{2}\omega_i\wedge\omega_j = \left(\operatorname{Id}+\mathcal{O}(\epsilon^{7/5})\right)_{ij}\otimes\operatorname{Vol}^{g^{q_j}}. \tag{11}$$

Setting $\mu=\operatorname{Vol}^{g^{q_j}}$ and $P=2\operatorname{Id}+\mathcal{O}(\epsilon^{7/5})$, we conclude that $\frac{1}{2}\Omega^{-2}\Lambda^{-1}\operatorname{Tf}(\omega\wedge\omega)$ is of order $\mathcal{O}(\Omega^{-2}\epsilon^{7/5})$ on the gluing regions and vanishes everywhere else. We claim

**Lemma 5.28.** *On the gluing regions, $\Omega^2$ and all its derivatives are of order $\epsilon^{-4/5}$ with respect to $g_{cf}$.*

*Proof.* Without loss of generality, we focus on a gluing region near a fixed point singularity $q_j$. Here, the function $\Omega^2$ is given by $e^{-2\rho_j}(h^{q_j}_{\epsilon})^{-1}$, where $\rho_j=\log(r_j)$. According to the product rule, the $C^k$ norm of $\Omega^2$ is bounded by the $C^k$ norm of $e^{-2\rho_j}$

and the $C^k$ norm of $(h_\epsilon^{q_j})^{-1}$. Because $\mathrm{d}\,\rho_j$ and its derivatives are uniformly bounded, $\|e^{-2\rho_j}\|_{C^k} = \mathcal{O}(e^{-2\rho_j})$, which is of order $\epsilon^{-4/5}$ on the gluing region $r_j \in (4\epsilon^{2/5}, 5\epsilon^{2/5})$.

We only need to show $(h_\epsilon^{q_j})^{-1}$ is uniformly bounded in all derivatives. By induction it is enough to show that $h_\epsilon^{q_j}$ is bounded below and $\mathrm{d}\,h_\epsilon^{q_j}$ is bounded in all derivatives. This lower bound is shown in Lemma 3.4. Furthermore, $\mathrm{d}\,h_\epsilon^{q_j}$ can be calculated explicitly and has order $\mathcal{O}(\epsilon r_j^{-1})$. On the gluing region, this order is $\mathcal{O}(\epsilon^{3/5})$. $\square$

We conclude that $\Omega^{-2}\Lambda^{-1}\,\mathrm{Tf}$ is of order $\mathcal{O}(\epsilon^{11/5})$ on the gluing region. These errors are measured with respect to the unweighted norm $g_{cf}^{q_j}$. Using the definition of the weighted norm[25] we can reintroduce the weights and conclude:

**Proposition 5.29.** *The constant term* $\frac{1}{2}\Omega^{-2}\Lambda^{-1}\,\mathrm{Tf}(\omega\wedge\omega)$ *is of order* $\mathcal{O}\left(\epsilon^{\frac{11-2\delta}{5}}\right)$ *with respect to* $(C_\delta^{k,\alpha}(M_{B,n})\cdot\omega)\otimes\mathbb{R}^3$.

### The linearised equation

Next we study the linearised term $L(\zeta) = \Omega^{-2}\Delta\zeta$, where $\Delta$ is the Hodge Laplacian on $\Omega^+(M_{B,n})$. As explained in the end of Section 2.2, the Weitzenböck formula implies

$$\sum_i L(u_i\cdot\omega_i) = (\Omega^{-2}\Delta^g u_i)\cdot\omega_i - 2\Omega^{-2}\nabla^g_{\nabla^g u_i}\omega_i,$$

for any twice differentiable set of functions $u_i$. We have chosen the domain of $L$, such that $\Omega^{-2}\Delta^g$ is an invertible operator with uniform bounded inverse. Hence $L$ have the same properties if $\Omega^{-2}\nabla^g_{\nabla^g u_i}\omega_i$ is sufficiently small. In this section we will prove this fact. We will only focus on the gluing region near a fixed point singularity $q_j$, because on the other gluing regions we have the same estimates and outside the gluing regions $\nabla^g\omega_i = 0$ due to the hyperkähler property. Also, we ignore any contributions from $\phi$, because $\phi$ vanishes on the gluing region.

Just as before, we first do the calculation in the weightless case and reintroduce the weights at the end. We will split the calculation of $2\Omega^{-2}\nabla^g_{\nabla^g u_i}\omega_i$ into several steps. First we calculate $\Omega^{-2}\nabla^g u_i$ and $\nabla^g\omega_i$ separately. The latter will be expanded into two parts, as

$$\nabla^g\omega_i = (\nabla^g - \nabla^{g^{q_j}})\omega_i + \nabla^{g^{q_j}}(\omega_i - \omega_i^{q_j}).$$

[25] Definition 5.2.

For the intermediary steps we will use the induced $C^{k,\alpha}_{cf}$ norms on the spaces of 2-forms and vector fields, which we write as $C^{k,\alpha}_{cf}(\Omega^2(M_{B,n}))$ and $C^{k,\alpha}_{cf}(TM_{B,n})$. At the end we estimate $2\Omega^{-2}\nabla^g_{\nabla^g u_i}\omega_i$ in terms of $C^{k,\alpha}_{\delta}(M_{B,n}) \cdot \omega$.

**Lemma 5.30.** *There exists a uniform constant $C > 0$, such that for any $u \in C^{k+2,\alpha}_{cf}(M_{B,n})$,*

$$\|\Omega^{-2}\nabla^g u\|_{C^{k+1,\alpha}_{cf}(TM_{B,n})} \leq C \cdot \|u\|_{C^{k+2,\alpha}_{cf}(M_{B,n})}.$$

*Proof.* Using $g_{cf} = \Omega^2 g^{q_j}$ and $g = g^{q_j} + \mathcal{O}(\epsilon^{11/5})$ on the gluing region, dualise $\Omega^{-2}\nabla^g u$ with respect to $g_{cf}$ to get

$$g_{cf}(\Omega^{-2}\nabla^g u, \ldots) = g^{q_j}(\nabla^g u, \ldots) = \mathrm{d}\, u + \mathcal{O}(\epsilon^{11/5})(\nabla^g u, \ldots).$$

Because $\Omega^2 = \mathcal{O}(\epsilon^{-4/5})$ on the gluing region, the difference between $\Omega^{-2}\,\nabla^g u$ and $\mathrm{d}\, u$ is of order $\mathcal{O}(\epsilon^{7/5})$, which proves the lemma. □

Next we will estimate $(\nabla^g - \nabla^{g^{q_j}})\omega_i$. We already know that $\omega_i$ is uniformly bounded with respect to $g^{q_j}$ and that it is of order $\mathcal{O}(\epsilon^{4/5})$ with respect to $g_{cf}$. Hence, we only need to estimate $\nabla^g - \nabla^{g^{q_j}}$.

**Lemma 5.31.** *With respect to $g_{cf}$, the term $\nabla^{g^{q_j}} - \nabla^g$ (and its derivatives) are of order $\mathcal{O}(\epsilon^{7/5})$ on the gluing region.*

*Proof.* Using the formula for the Christoffel symbols in Riemann normal coordinates of $g_{cf}$, the error term is of the form

$$\frac{1}{2} g^{im}_{q_j} \left[ \frac{\partial}{\partial x^l}\left(g_{mk} - g^{q_j}_{mk}\right) + \ldots \right] + \frac{1}{2}(g^{im} - g^{im}_{q_j}) \left[ \frac{\partial g_{mk}}{\partial x^l} + \ldots \right].$$

The difference between $g$ and $g^{q_j}$ is of order $\mathcal{O}(\epsilon^{11/5})$. Also, $g^{-1} = \Omega^2 g^{-1}_{cf} = \mathcal{O}(\epsilon^{-4/5})$ and hence $\nabla^{g^{q_j}} - \nabla^g$ simplifies to

$$\mathcal{O}(\epsilon^{7/5}) + \mathcal{O}(\epsilon^{11/5}) \cdot \left[ \frac{\partial\, \Omega^{-2} \cdot g^{cf}_{mk}}{\partial x^l} + \ldots \right].$$

This is of order $\mathcal{O}(\epsilon^{7/5})$. □

**Lemma 5.32.** *On the gluing region, the term $\nabla^{g^{q_j}}(\omega_i - \omega_i^{q_j})$ is of order $\mathcal{O}(\epsilon^{11/5})$ with respect to $C^{k,\alpha}_{cf}(\Omega^2(M_{B,n}))$.*

*Proof.* According to Theorem 3.23, on the gluing region $\omega_i - \omega_i^{q_j}$ is of order $\mathcal{O}(\epsilon^{11/5})$ with respect to $g^{q_j}_{cf}$. In terms of the Riemann normal coordinates $\{x_k\}$ for $g^{q_j}_{cf}$, we calculate

$$\begin{aligned}\left[\nabla^{g^{q_j}}_{\partial x_i}(\omega_i - \omega_i^{q_j})\right](\partial x_k, \partial x_l) =& \mathcal{L}_{\partial x_i}\left[(\omega_i - \omega_i^{q_j})(\partial x_k, \partial x_l)\right] \\ &+ (\omega_i - \omega_i^{q_j})(\nabla^{g^{q_j}}_{\partial_i}\partial x_k, \partial x_l) + (\omega_i - \omega_i^{q_j})(\nabla^{g^{q_j}}_{\partial_i}\partial x_k, \partial x_l) \\ =& \mathcal{O}(\epsilon^{11/5}) + \mathcal{O}(\epsilon^{11/5}) \cdot \|\nabla^{g^{q_j}}_{\partial x_i}\partial x_k\|_{cf}.\end{aligned}$$

By the Koszul formula, $\nabla^{g^{q_j}}_{\partial x_i}\partial x_k$ is bounded by $\mathrm{d}\log\Omega$, which is uniformly bounded. Therefore, $\nabla^{g^{q_j}}(\omega_i - \omega_i^{q_j})$ and all its derivatives are of order $\mathcal{O}(\epsilon^{11/5})$. □

We conclude that on the gluing region $\Omega^{-2}\nabla^g_{\nabla^g u_i}\omega_i$ is of order $\mathcal{O}(\epsilon^{11/5})$ with respect to the induced norm of $g_{cf}$ on $\Omega^+(M_{B,n})$. However, in the beginning of this section we equipped $\Omega^+$ with the $C^{k+2,\alpha}_{cf}(M_{B,n}) \cdot \omega$ norm instead. To convert to this norm, we project $\Omega^{-2}\nabla^g_{\nabla^g u_i}\omega_i$ to the approximately orthogonal basis $\{\omega_i\}$, whose basis elements satisfy $\|\omega_i\|_{cf} = \mathcal{O}(\epsilon^{4/5})$. Therefore, with respect to the norm $C^{k+2,\alpha}_{cf}(M_{B,n})\cdot\omega$, the form $\Omega^{-2}\nabla^g_{\nabla^g u_i}\omega_i$ is of order $\epsilon^{7/5}$. The only thing left is to reintroduce the weights:

**Proposition 5.33.** *The operator $2\Omega^{-2}\nabla^g_{\nabla^g(\dots)}\omega_i$ between $C^{k+2,\alpha}_\delta(M_{B,n})$ (or $C^{k+2,\alpha}_\delta(M_{B,n}) \oplus \mathbb{R}\phi$ when $B \neq \mathbb{R}^3$) and $C^{k+1,\alpha}_\delta(M_{B,n}) \cdot \omega$ is a bounded operator with an operator norm of order $\mathcal{O}(\epsilon^{7/5})$. In particular, for sufficiently small $\epsilon > 0$, the map $\Omega^{-2}\Delta_{Hodge}$ is an isomorphism with uniform bounded inverse.*

*Proof.* Let $u_i$ in the domain and consider $2\Omega^{-2}\nabla^g_{\nabla^g u_i}\omega_i$. In local normal coordinates this operator can be written as

$$2\Omega^{-2}\nabla^g_{\nabla^g u_i}\omega_i = \left(A^{jk}\frac{\partial u_i}{\partial x_k} + B^j \cdot u_i\right)\omega_j,$$

where $A^{jk}$ and $B^j$ are smooth functions of order $\mathcal{O}(\epsilon^{7/5})$ with respect to $g_{cf}$. To estimate these in the $C^{k+1,\alpha}_\delta(M_{B,n})\cdot\omega$ norm, we need to calculate $e^{-\delta\rho}\left(A^{jk}\frac{\partial u_i}{\partial x_k} + B^j \cdot u_i\right)$.

This can be written as

$$e^{-\delta\rho}\left(A^{jk}\frac{\partial u_i}{\partial x_k}+B^j\cdot u_i\right)=A^{jk}\frac{\partial(e^{-\delta\rho}u_i)}{\partial x_k}+\left(\delta\, A^{jk}\frac{\partial\rho}{\partial x_k}+B^j\right)\cdot e^{-\delta\rho}u_i.$$

Using $e^{-\delta\rho}u_i\in C^{k+2,\alpha}_{cf}(M_{B,n})$, we conclude

$$\|2\Omega^{-2}\nabla^g_{\nabla^g u_i}\omega_i\|_{C^{k+1,\alpha}_\delta(M_{B,n})\cdot\omega}\leq\mathcal{O}(\epsilon^{7/5})\cdot\|u_i\|_{C^{k+2,\alpha}_\delta(M_{B,n})},$$

which proves the first statement. The second statement follows from the Weitzenböck formula. □

### The non-linear term

Finally, we study the non-linear part $N(\zeta)=2\Omega^{-2}\,\mathrm{Tf}((\mathrm{d}\,\mathrm{d}^*\zeta)^2)$. Again, we do this in multiple steps. First, we estimate $\mathrm{d}\,\mathrm{d}^*\zeta$ in terms of $C^{k,\alpha}_{cf}(\Omega^2(M_{B,n}))$. Secondly, we work out $N(\zeta)-N(\xi)$ using the product rule for Hölder norms, which yields an explicit error. Finally, we calculate this error on each region separately.

**Lemma 5.34.** *Let $\zeta\in C^{k+2,\alpha}_{cf}(M_{B,n})\cdot\omega$ (or $\zeta\in(C^{k+2,\alpha}_{cf}(M_{B,n})\oplus\mathbb{R}\phi)\cdot\omega$ when $B\neq\mathbb{R}^3$). There exists a constant $C>0$, independent of $\zeta$ and $\epsilon$, such that*

$$\|\,\mathrm{d}\,\mathrm{d}^*\zeta\|_{C^{k,\alpha}_\delta(\Omega^2(M_{B,n}))}\leq C\cdot\|\zeta\|_{C^{k+2,\alpha}_\delta(M_{B,n})\cdot\omega}$$

*when $B=\mathbb{R}^3$ or*

$$\|\,\mathrm{d}\,\mathrm{d}^*\zeta\|_{C^{k,\alpha}_\delta(\Omega^2(M_{B,n}))}\leq C\cdot\|\zeta\|_{(C^{k+2,\alpha}_\delta(M_{B,n})\oplus\mathbb{R}\phi)\cdot\omega}$$

*when $B\neq\mathbb{R}^3$.*

*Proof.* Consider the case when $\delta=0$, and expand $\zeta$ into $\zeta=\sum_i u_i\cdot\omega_i$. Using that $\omega_i$ is closed and self-adjoint with respect to $g$,

$$\mathrm{d}\,\mathrm{d}^*\zeta=-\sum_i\mathrm{d}*^g\mathrm{d}*^g(u_i\cdot\omega_i)=-\sum_i\mathrm{d}*^g\mathrm{d}\,u_i\wedge\omega_i=-\sum_i\mathrm{d}\,\iota_{\nabla^g u_i}\omega_i.$$

The exterior derivative is a bounded linear map between $C^{k+1,\alpha}_{cf}(\Omega^1(M_{B,n}))$ and $C^{k,\alpha}_{cf}(\Omega^2(M_{B,n}))$, and hence it is sufficient to show $\iota_{\nabla^g u_i}\omega_i$ and its derivatives are uniformly bounded. By a similar argument as in Lemma 5.30, $\nabla^g u_i$ is of order

$\mathcal{O}(\Omega^2)$. Because $\phi = \rho$ on the asymptotic region, the same is true when $u_i$ is a multiple of $\phi$. Finally, recall that $\omega$ and its derivatives are of order $\Omega^{-2}$ with respect to $g_{cf}$ and therefore $\iota_{\nabla^g u_i}\omega_i$ is uniformly bounded.

Just as in Proposition 5.33, the estimate does not change when we reintroduce the weights. Hence $\mathrm{d}\,\mathrm{d}^*$ is a uniformly bounded operator. □

For any $f, g \in C^{k,\alpha}_{cf}(M_{B,n})$, the product rule of Hölder norms implies $f \cdot g \in C^{k,\alpha}_{cf}(M_{B,n})$ and $\|f \cdot g\|_{C^{k,\alpha}_{cf}(M_{B,n})} \leq C\|f\|_{C^{k,\alpha}_{cf}(M_{B,n})} \cdot \|g\|_{C^{k,\alpha}_{cf}(M_{B,n})}$ for some uniform constant $C$. This implies that the wedge product can be viewed as the bounded linear map

$$\wedge\colon C^{k,\alpha}_{cf}(\Omega^2(M_{B,n})) \times C^{k,\alpha}_{cf}(\Omega^2(M_{B,n})) \to C^{k,\alpha}_{cf}(\Omega^4(M_{B,n})).$$

With this version of the Hölder product rule, we can prove the non-linear condition for the inverse function theorem.

**Proposition 5.35.** *Let* $N(\zeta) = 2\Omega^{-2}\Lambda^{-1}\,\mathrm{Tf}(\mathrm{d}\,\mathrm{d}^*\zeta \wedge \mathrm{d}\,\mathrm{d}^*\zeta)$. *There exists a* $q > 0$ *of order* $\mathcal{O}(\epsilon^{\delta-2})$ *such that for any* $\zeta, \xi \in (C^{k+2,\alpha}_{\delta}(M_{B,n})\cdot\omega)\otimes\mathbb{R}^3$ *(or* $(C^{k+2,\alpha}_{\delta}(M_{B,n})\cdot\omega \oplus \mathbb{R}\phi) \otimes \mathbb{R}^3$ *when* $B \neq \mathbb{R}^3$*),*

$$\|N(\zeta) - N(\xi)\|_{(C^{k,\alpha}_{\delta}(M_{B,n})\cdot\omega)\otimes\mathbb{R}^3} \leq q \cdot \|\zeta + \xi\| \cdot \|\zeta - \xi\|,$$

*where* $\|\zeta \pm \xi\|$ *is measured with the* $(C^{k+2,\alpha}_{\delta}(M_{B,n})\cdot\omega)\otimes\mathbb{R}^3$ *or* $(C^{k+2,\alpha}_{\delta}(M_{B,n})\cdot\omega \oplus \mathbb{R}\phi) \otimes \mathbb{R}^3$ *norm respectively.*

*Proof.* Using the 'identity' $a^2 - b^2 = (a+b)(a-b)$, the expression of for $N(\zeta) - N(\xi)$ can be rewritten as

$$N(\zeta) - N(\xi) = 2\Omega^{-2}\Lambda^{-1}\,\mathrm{Tf}(\mathrm{d}\,\mathrm{d}^*(\zeta + \xi) \wedge \mathrm{d}\,\mathrm{d}^*(\zeta - \xi)).$$

Using Lemma 5.34 and the product rule, $N(\zeta) - N(\xi)$ can be estimated by

$$N(\zeta) - N(\xi) = \mathcal{O}(2e^{\delta\rho}\Omega^{-2}\Lambda^{-1}\,\mathrm{Tf}(\mathrm{Vol}^{g_{cf}})) \cdot \|\zeta + \xi\| \cdot \|\zeta - \xi\|.$$

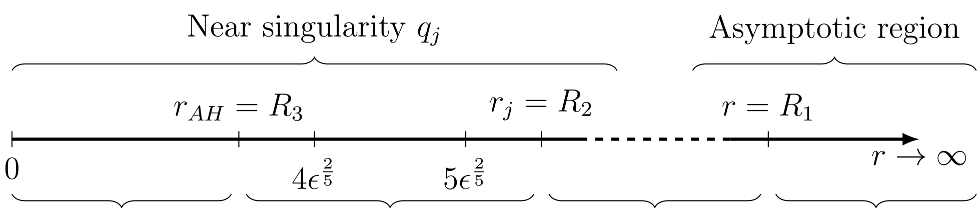
Recall that the map Tf projects the space of 3 by 3 matrices to its symmetric traceless subspace. This projection is uniformly bounded, and hence

$$N(\zeta) - N(\xi) = \mathcal{O}(2e^{\delta\rho}\Omega^{-2}\Lambda^{-1}(\mathrm{Vol}^{g_{cf}})) \cdot \|\zeta + \xi\| \cdot \|\zeta - \xi\|.$$

Using Equation 11 and that $g$ is hyperkähler outside the gluing region, we estimate the inverse of $\Lambda$, which yields

$$N(\zeta) - N(\xi) = \mathcal{O}(e^{\delta\rho}\Omega^{2}) \cdot \omega \cdot \|\zeta + \xi\| \cdot \|\zeta - \xi\|.$$

We conclude that $q$ must be of order $\mathcal{O}(e^{\delta\rho}\Omega^2)$. We calculate $\mathcal{O}(e^{\delta\rho}\Omega^2)$ explicitly for each region of $M_{B,n}$, which are given in Definition 5.1. We summarise the estimates in the following table[26]:




| | | | | |
|---|---|---|---|---|
| $\Omega$ | $\frac{\sqrt{1+\epsilon\alpha_j}}{\epsilon}$ | $r_j^{-1}(h_\epsilon^{q_j})^{-\frac{1}{2}}$ | 1 | (Def. 4.7) |
| $\rho$ | $\log\left(\frac{\epsilon}{1+\epsilon\alpha_j}R_3\right)$ | $\log(r_j)$ | 1 | (Def. 4.9) |
| $\mathcal{O}(e^{\delta\rho}\Omega^2)$ | $\mathcal{O}(\epsilon^{\delta-2})$ | $\mathcal{O}(r_j^{\delta-2}) \leq \mathcal{O}(\epsilon^{\delta-2})$ | $\mathcal{O}(1)$ | $\mathcal{O}(1)$ |

The parameter $q$ attains its largest value inside the bubbles, and hence $q = \mathcal{O}(\epsilon^{\delta-2})$. □

As explained in Section 2.2, our goal is to solve the equation

$$\frac{1}{2}\Lambda^{-1}\,\mathrm{Tf}(\omega \wedge \omega) + \Delta\zeta + 2\Lambda^{-1}\,\mathrm{Tf}(\mathrm{d}\,\mathrm{d}^*\zeta \wedge \mathrm{d}\,\mathrm{d}^*\zeta) = 0. \tag{6}$$

According to Proposition 5.29, the constant part $F(0)$ is of order $\mathcal{O}(\epsilon^{\frac{11-2\delta}{5}})$. According to Proposition 5.33, the linearised operator is invertible with uniform bounded inverse. By Proposition 5.35, the non-linear part satisfies

$$\|N(\zeta) - N(\xi)\|_{(C^{k,\alpha}_\delta(M_{B,n})\cdot\omega)\otimes\mathbb{R}^3} \leq \mathcal{O}(\epsilon^{\delta-2}) \cdot \|\zeta + \xi\| \cdot \|\zeta - \xi\|.$$

[26] Again the estimates for a non-fixed point singularity $p_i$ are the same as the estimates for a fixed point singularity $q_j$ and we can ignore them.

Therefore, Theorem 2.15 can be applied if

$$\epsilon^{\frac{11-2\delta}{5}} \leq \mathcal{O}(\epsilon^{2-\delta}).$$

This is indeed true for sufficiently small $\epsilon$, and hence:

**Proposition 5.36.** *For sufficiently small $\epsilon > 0$, there exists a triple of $\zeta_i \in C^{k,\alpha}_\delta(M_{B,n}) \cdot \omega$ (or $\zeta \in (C^{k,\alpha}_\delta(M_{B,n}) \oplus \mathbb{R}\phi) \cdot \omega$ when $B \neq \mathbb{R}^3$), such that*

$$\omega_i + 2\,\mathrm{d}\,\mathrm{d}^*\,\zeta_i$$

*is an orthonormal triple of closed 2-forms, and the norm of $\zeta_i$ is of order $\mathcal{O}(\epsilon^{\frac{11-2\delta}{5}})$.*

### Higher regularity

At last, we need to show $\omega + 2\,\mathrm{d}\,\mathrm{d}^*\,\zeta$ is smooth. For this we use a bootstrapping argument. Namely, we know that $\zeta$ is a $C^{k+2,\alpha}$ solution of Equation 6, which is of the form

$$F(0) + L(\zeta) + N(\mathrm{d}\,\mathrm{d}^*\,\zeta, \mathrm{d}\,\mathrm{d}^*\,\zeta) = 0.$$

Hence any partial derivative $\dot{\zeta} \in C^{k+1,\alpha}$ must satisfy an equation of the form

$$\begin{aligned}\dot{F}(0) + \dot{L}(\zeta) + L(\dot{\zeta}) + \dot{N}(\mathrm{d}\,\mathrm{d}^*\,\zeta, \mathrm{d}\,\mathrm{d}^*\,\zeta)&\\ + N(\mathrm{d}\,\mathrm{d}^*\,\dot{\zeta}, \mathrm{d}\,\mathrm{d}^*\,\zeta) + N(\mathrm{d}\,\mathrm{d}^*\,\zeta, \mathrm{d}\,\mathrm{d}^*\,\dot{\zeta}) &= 0,\end{aligned}$$

where the dot denotes the partial derivatives of the coefficients. Therefore, there is some $\tilde{F} \in C^{k,\alpha}_{loc}$ such that

$$L(\dot{\zeta}) + N(\mathrm{d}\,\mathrm{d}^*\,\dot{\zeta}, \mathrm{d}\,\mathrm{d}^*\,\zeta) + N(\mathrm{d}\,\mathrm{d}^*\,\zeta, \mathrm{d}\,\mathrm{d}^*\,\dot{\zeta}) = \tilde{F}.$$

We claim that the operator

$$L + N(\mathrm{d}\,\mathrm{d}^* \ldots, \mathrm{d}\,\mathrm{d}^*\,\zeta) + N(\mathrm{d}\,\mathrm{d}^*\,\zeta, \mathrm{d}\,\mathrm{d}^* \ldots) \tag{12}$$

is a strictly elliptic operator, because the operator norm of $N(\mathrm{d}\,\mathrm{d}^*\,\zeta, \mathrm{d}\,\mathrm{d}^* \ldots)$ is arbitrary small. Indeed, using a similar argument as in Proposition 5.35 and Lemma

5.34, one can show that

$$\|N(\mathrm{d}\,\mathrm{d}^*\,\zeta, \mathrm{d}\,\mathrm{d}^*\ldots)\|_{op} = \mathcal{O}(\epsilon^{\delta-2}) \cdot \|\zeta\|_{C^{k+2,\alpha}}.$$

By Proposition 5.36, we know $\|\zeta\|_{C^{k+2,\alpha}} = \mathcal{O}(\epsilon^{\frac{11-2\delta}{5}})$, and hence

$$\|N(\mathrm{d}\,\mathrm{d}^*\,\zeta, \mathrm{d}\,\mathrm{d}^*\ldots)\|_{op} = \mathcal{O}(\epsilon^{\frac{1}{5}+\frac{3}{5}\delta}).$$

For sufficiently small $\epsilon$, the operator in Equation 12 is elliptic. Using the local Schauder estimate, $\dot{\zeta}$ must be an element of $C^{k+2,\alpha}$ and so $\zeta \in C^{k+3,\alpha}$. Using induction on $k$, we conclude:

**Theorem 5.37.** *For sufficiently small $\epsilon > 0$, there exists a smooth triple of $\zeta_i \in C^{k,\alpha}_\delta(M_{B,n}) \cdot \omega$ (or $\zeta_i \in (C^{k,\alpha}_\delta(M_{B,n}) \oplus \mathbb{R}\phi) \cdot \omega$ when $B \neq \mathbb{R}^3$), such that*

$$\omega_i + 2\,\mathrm{d}\,\mathrm{d}^*\,\zeta_i$$

*is a hyperkähler triple.*

We finally show the main result of this thesis.

**Theorem 1.1.** *Let $L \subset \mathbb{R}^3$ be a lattice of rank one or two and consider the $\mathbb{Z}_2$ action on $\mathbb{R}^3/L$ that is induced by the antipodal map on $\mathbb{R}^3$. Let $\{p_i\}$ be a configuration of $n$ distinct points in $(\mathbb{R}^3/L - \mathrm{Fix}(\mathbb{Z}_2))/\mathbb{Z}_2$. Suppose that $n \leq 4$ when $\mathbb{R}^3/L \simeq \mathbb{R}^2 \times S^1$ and $n \leq 8$ when $\mathbb{R}^3/L \simeq \mathbb{R} \times T^2$. Then, there exists an $\epsilon_0 > 0$, such that for all $0 < \epsilon < \epsilon_0$ there exist a gravitational instanton $(M_{\mathbb{R}^3/L,n}, g_\epsilon)$ with the following properties:*

1. *For each fixed point of the $\mathbb{Z}_2$ action on $\mathbb{R}^3/L$, there is a compact set $K \subset M_{\mathbb{R}^3/L,n}$, such that $\epsilon^{-2}g_\epsilon$ approximates the Atiyah-Hitchin metric on $K$ as $\epsilon \to 0$.*

2. *For each $i \in \{1, \ldots, n\}$, there is a compact set $K_i \subset M_{\mathbb{R}^3/L,n}$ such that $\epsilon^{-2}g_\epsilon$ approximates the Taub-NUT metric on $K_i$ as $\epsilon \to 0$.*

3. *Away from the singularities, the manifold collapses to $(\mathbb{R}^3/L)/\mathbb{Z}_2$ with bounded curvature as $\epsilon$ converges to zero.*

4. *Depending on the lattice and $n$, the asymptotic metric can be classified as*

- *ALG*-*$I^*_{4-n}$ *when* $\dim L = 1$ *and* $n < 4$*,*
- *ALG*$_{\frac{1}{2}}$ *when* $\dim L = 1$ *and* $n = 4$*,*
- *ALH**$_{8-n}$ *when* $\dim L = 2$ *and* $n < 8$*,*
- *ALH when* $\dim L = 2$ *and* $n = 8$*.*

*Proof.* Given the data in the theorem, we constructed in Chapter 3 a 4-manifold $M_{B,n}$ and a 1-parameter family of closed definite triples $\omega$ that are approximately hyperkähler. By theorem 5.37, there exists a smooth $\zeta \in \Omega^+(M_{B,n}) \otimes \mathbb{R}^3$ such that $\omega + 2\,\mathrm{d}\,\mathrm{d}^*\,\zeta$ is a smooth hyperkähler triple, which induces a hyperkähler metric on $M_{B,n}$. Moreover, our genuine gravitational instanton differs from our approximate solution with an error of $\mathcal{O}(\epsilon^{\frac{11-2\delta}{5}})$. Hence, for sufficiently small $\epsilon$, properties 1 to 4 are satisfied. $\square$

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